Some Results on Best Approximations And Farthest Points

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CERTIFICATE

I am to certify that the research work embodied in the dissertation "Some Results on Best Approximations and Farthest Points" by B.B. Panda, a Ph.D. student of this department, has been carried out under my supervision and that it has not been submitted elsewhere for any degree.

O. P. Kapoor

May 1, 1974. (O.P. KAPOOR)

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May 1, 1974.

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CONTENTS

SYNOPSIS

The continuity of metric projections and the singletonness of sets having unique farthest point property have been studied by many mathematicians. The purpose of this thesis is to continue the study of these problems. It is familiar that the class of strictly convex spaces with the Efimov Stechkin property (the so called E-spaces) provides a satisfactory setting for convex best approximation problems. In this thesis we introduce another class of Banach spaces which we call "spaces with property (M)" and consider in such spaces certain problems regarding continuity of metric projections and farthest point maps. It appears that such spaces may turn out to be nice for problems dealing with farthest points.

There are five chapters in this dessertation.

Chapter 1 contains some basic preliminaries, a general outline of the thesis, and a brief introduction to the recent developments in the field of nearest and farthest points of sets.

In chapter 2, we have considered the following properties :

A normed linear space X is said to possess property

(M) if whenever $x, g_n \in X$, $||x|| = 1$, $||g_n|| \leq 1$, and $||x + g_n|| \to 2$, then the sequence $\{g_n\}$ is compact ;

(WM) if whenever $x_o, x_n \varepsilon X$, $||x_o|| = 1 = ||x_n||$, $f_o \varepsilon X^*$, $||f_o|| = 1$, $f_o(x_o) = 1$, and $||x_o + x_n|| \to 2$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $f_o(x_{n_i}) \to 1$;

(S) if whenever $\phi_n \varepsilon X^*$, $||\phi_n|| \leq 1$, $x_o \varepsilon X$, $||x_o|| = 1$, and $\phi_n(x_o) \to 1$, then the sequence $\{\phi_n\}$ is compact;

(h) if whenever $x_n, x_o \varepsilon X$, $x_n \rightharpoonup x_o$, $||x_n|| \to ||x_o||$, then $x_n \to x_o$;

(Efimov Stechkin) every weakly sequentially closed set is approximatively compact.

We have studied the inter-relationship among these properties and have shown that in a reflexive space there is complete duality between property (S) and Efimov Stechkin property. It is also shown that in a space with property (M), if $x \neq \theta$, $||g_n|| \leq 1$, and $||x-g_n|| \to 1 + ||x||$, or if $0 < ||x|| \leq 1$, $||g_n|| \geq 1$, $||x-g_n|| \to 1- ||x||$, then the sequence $\{g_n\}$ is compact. Examples have been given to show that property (M) $\not\Rightarrow$ local uniform convexity of the norm, and Efimov Stechkin property $\not\Rightarrow$ property (M).

In the first part of chapter 3, we have generalized a result of E. Asplund which states that if K is a bounded and closed subset of a reflexive, locally uniformly convex Banach space X, then it admits farthest points from points of a dense subset of X. We have shown that this result remains true if either (i) X is reflexive and has property (M), or (ii) K is bounded and weakly sequentially closed and X is reflexive with property (WM). In the second part

of this chapter, we have introduced a concept of M-compact set, namely, a set $K \subset X$ is M-compact, if whenever $x \in X$ and $\{g_n\} \subset K$ with $||x-g_n|| \to F_K(x) = \sup \{||x-y|| : y \in K\}$, then the sequence $\{g_n\}$ is compact in K. Such a sequence $\{g_n\}$ is said to be a maximizing sequence for x. Clearly every compact set is M-compact. We have constructed noncompact M-compact sets and have studied some of their properties. We have also obtained a formula for the subdifferential $\partial F_K(x)$ at a point $x \in X$, where K is an M-compact set. In the third part of this chapter, we have shown that in a locally uniformly convex space X, every bounded set having unique farthest point property supports a farthest point map whose domain of continuity is dense in X. We have obtained some relationships among the continuity of the farthest point map, the Gateaux-differentiability of the function F_K, and the compactness of maximizing sequences in K. In the last part of this chapter, we have shown that in a normed linear space "admitting centres", a set having unique farthest point property is a singleton, provided the associated farthest point map satisfies a mild continuity condition.

In Chapter 4, we have proved that in a space with property (M), the approximative compactness of a Chebyshev set is equivalent to the continuity of the associated metric projection. That an approximatively compact Chebyshev set supports a continuous metric projection has been shown by I. Singer. An example has been given to show that in an arbitrary space, the continuity of the metric projection may not imply the approximative compactness of the supporting set. It is also proved

that in a space X with property (M), every Chebyshev set supports a metric projection whose domain of continuity is dense in X. This result has been further improved in locally uniformly convex and uniformly convex Banach spaces by relaxing the Chebyshev property of the supporting set.

Chapter 5 deals with some geometrical and topological properties of equidistant sets considered by G.K. Kalish and E.G. Straus. It is proved, for example, that if for each nonzero x in a normed linear space X the set of points equidistant from x and $-x$ is convex, then X must be an inner product space. It is proved that every equidistant set in an ℓ^p-space $(1 < p < \infty)$ is bounded weakly closed. Thus these spaces satisfy the P_2-property considered by V. Klee. On the other hand, no equidistant set in $L^p(\mu)$ $(1< p < \infty,\ p \neq 2;\ \mu$ a separable nonatomic measure) and c_0 is weakly sequentially closed.

CHAPTER - I

PRELIMINARIES AND BASIC CONCEPTS

We begin by recalling some basic definitions, notation, and theorems that will be needed throughout the thesis. Any term not specifically defined can be found in [24] .

For all normed linear spaces discussed in this thesis the scalars are assumed to be real and their field is denoted by R. A normed linear space is denoted by X and its conjugate space by X^*. The unit ball and the unit sphere in X are denoted by $U(X)$ and $S(X)$ respectively. The symbol $B[x,r]$ denotes the closed ball of radius r with centre at x. θ stands for the null element of the space.

1.1 Various Norms . We shall now consider various types of norms that will be freely used in the main body of the thesis. The results of this section can be found in [16] , [19] , [55] and [62].

Definition 1.1.1 . A norm (or a normed linear space) is said to be uniformly convex if and only if, given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta(\varepsilon) \quad \text{whenever } \|x-y\| \geq \varepsilon \text{ , and } \|x\|=\|y\|= 1.$$

Definition 1.1.2 . A norm (or a normed linear space) is said to be locally uniformly convex if and only if, given $\varepsilon > 0$ and an element x with $||x|| = 1$, there exists $\delta(\varepsilon, x) > 0$ such that

$$||\frac{x+y}{2}|| \leq 1-\delta(\varepsilon) \quad \text{whenever} \quad ||x-y|| \geq \varepsilon \text{ and } ||y|| = 1.$$

Definition 1.1.3 . A norm (or a normed linear space) is said to be strictly convex if and only if, $||x+y|| = ||x|| + ||y||$ implies $x = ty$, $t > 0$, whenever $x \neq \theta$ and $y \neq \theta$.

It is clear from the definitions that uniform convexity implies local uniform convexity, and local uniform convexity implies strict convexity.

Theorem 1.1.4 . A uniformly convex space is reflexive.

Let X be a normed linear space. Consider the quotient

$$\Delta(x,y,t) = \frac{||x+ty|| - ||x||}{t}, \tag{1.1.1}$$

where $x,y \in X$ and t is a real number. It can be seen [24] that

$$\text{for} \quad t_1 \geq t_2 > 0, \ \Delta(x,y,t_1) \geq \Delta(x,y,t_2) \geq -||y|| , \tag{1.1.2}$$

$$\text{for} \quad t_1 \leq t_2 < 0, \ \Delta(x,y,t_1) \leq \Delta(x,y,t_2) \leq ||y|| , \tag{1.1.3}$$

$$\text{and for} \quad t > 0 , \ \Delta(x,y,t) + \Delta(x,-y,t) \geq 0 . \tag{1.1.4}$$

As a consequence we have the following :

$$\Delta(x,y,-t) \leq \tau^-(x;y) \leq \tau^+(x;y) \leq \Delta(x,y,t), \ t > 0, \tag{1.1.5}$$

where τ^{+} and τ^{-} are the right and the left limits of the quotient $\Delta(x,y,t)$ as $t \to 0$. If $\tau^{+}(x;y) = \tau^{-}(x;y)$ for $y \in X$, the common limit

$$G(x;y) = \lim_{t \to 0} \frac{||x+ty|| - ||x||}{t} \tag{1.1.6}$$

is called the Gateaux derivative of the norm at x in the direction of y.

Definition 1.1.5 . A norm (resp. a normed linear space X) is said to be smooth if it (resp. its norm) is G-differentiable at every point of the unit sphere $S(X)$. The norm is said to be strongly smooth if the convergence in (1.1.6) is uniform in $y \in S(X)$. The derivative in this case is called the Frechet derivative of the norm. If the convergence in (1.1.6) is uniform both in x and y, where $x \in S(X)$ and $y \in S(X)$, the norm (or the space) is said to be uniformly smooth.

Theorem 1.1.6 . If the conjugate space X^* is strictly convex (smooth), the space X is smooth (strictly convex). This duality is complete in reflexive Banach spaces.

Theorem 1.1.7 . If X is reflexive, locally uniformly convex, then the norm in X^* is strongly smooth.

Theorem 1.1.8 . If X^* is locally uniformly convex, then the norm is strongly smooth in X .

Theorem 1.1.9 . A normed linear space is uniformly smooth (resp. uniformly convex) if and only if its conjugate space is uniformly convex (resp. uniformly smooth).

Definition 1.1.10 . The norms $||\cdot||$ and $||\cdot||_1$ are equivalent if and only if, for any sequence $\{x_n\}$, $\lim ||x_n|| = 0$ if and only if $\lim ||x_n||_1 = 0$.

It can be shown that a n.a.s.c. for $||\cdot||$ and $||\cdot||_1$ to be equivalent is that there exist numbers a,b with $0 < a \leq b < \infty$ such that

$$a||x|| \leq ||x||_1 \leq b||x|| \text{ for all } x. \tag{1.1.7}$$

1.2 Weak Topologies. The results of this section can be obtained in [24] .

Definition 1.2.1 . The weak topology on X is the topology obtained by taking as base all sets of the form

$$N(x;A,\varepsilon) = \{y \in X : |x^*(x) - x^*(y)| < \varepsilon,\ x^* \in A\},$$

where $x \in X$, A is a finite subset of X^*, and $\varepsilon > 0$.

Definition 1.2.2 . The weak* topology on X^* is the topology obtained by taking as base all sets of the form

$$N(x^*,A,\varepsilon) = \{y^* \in X^* : |x^*(x) - y^*(x)| < \varepsilon,\ x \in A\},$$

where $x^* \in X^*$, A is a finite subset of X, and $\varepsilon > 0$.

It is easy to see that if X is a reflexive Banach space, then both the weak and weak* topologies on X^* are the same.

<u>Theorem</u> 1.2.7 . A set in a reflexive space is relatively weakly sequentially compact if and only if it is bounded.

1.3 <u>Convex Sets</u> . Let X be a linear space and K be a subset of X. Then K is said to be <u>convex</u> if for any two points $x_1, x_2 \varepsilon K$, $\lambda x_1 + (1-\lambda)x_2 \varepsilon K$ for all λ with $0 \leq \lambda \leq 1$. For an arbitrary set K, the <u>convex hull</u> of K, denoted by $co(K)$, is the intersection of all convex sets containing K. The closure of the convex hull of K is denoted by $\overline{co}(K)$.

<u>Lemma</u> 1.3.1 . Let K_1, K_2 be two sets in a linear topological space. If the closed convex hulls of K_1 and K_2 are compact, then $\overline{co}\,(K_1 \cup K_2) = co(\,\overline{co}\,(K_1) \cup \overline{co}\,(K_2))$.

<u>Theorem</u> 1.3.2 . (Mazur) Let X be a Banach space, and let $K \subset X$ be compact. Then $\overline{co}(K)$ is compact.

<u>Theorem</u> 1.3.3 . (Strong Separation Theorem) If K_1 and K_2 are disjoint closed convex sets of a locally convex linear topological space X, and if K_1 is compact, then there exist constants c and ε , $\varepsilon > 0$, and a continuous linear functional f on X, such that

$$Rf(K_2) \leq c-\varepsilon < c \leq Rf(K_1).$$

<u>Definition</u> 1.3.4 . In a linear space X, a point x in a convex set K is called an extreme point of K if $x_1, x_2 \varepsilon K$, $0 < \lambda < 1$ and $x = \lambda x_1 + (1-\lambda)\, x_2$ imply that $x_1 = x_2 = x$. The set of extreme points of K is denoted by $Ext(K)$.

<u>Theorem</u> 1.3.5 . (Krein-Milman) Let K be a compact convex subset of a locally convex linear topological space X. Then $K = \overline{co}\,(\mathrm{Ext}\,(K))$.

1.4 <u>Convex Functions</u> . [38] We say that a function f, defined on a linear space X, is a proper convex function if it is not identically $+\infty$ and

$$f(tx+(1-t)y) \leq t\,f(x) + (1-t)\,f(y),$$

whenever $x,y \in X$ and $0 \leq t \leq 1$. Such a function will be denoted by $f \in \mathrm{conv}(X)$. The <u>effective domain</u> of f is the set

$$\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}.$$

<u>Theorem</u> 1.4.1 . Let X be a linear space and $f \in \mathrm{conv}(X)$. Then if $x_o \in \mathrm{dom}(f)$,

$$f'(x_o;x) = \lim_{t \to +0} \frac{f(x_o+tx) - f(x_o)}{t} \tag{1.4.1}$$

exists in $[-\infty,\infty]$ for every $x \in X$.

<u>Remark</u> 1.4.2 . The quotient $f(x_o+tx) - f(x_o)/t$ is a non-decreasing function of t for $t > 0$.

<u>Theorem</u> 1.4.3 . Let f be a finite convex function on a linear space X. Then $f'(x_o;\,.)$ is a (finite) sublinear function on X, for all $x_o \in X$.

<u>Corollary</u> 1.4.4 . Let f be as in the Theorem 1.4.3. Then

$$\lim_{t \to -0} \frac{f(x_o+tx) - f(x_o)}{t} = -f'(x_o,-x) \leq f'(x_o;x),$$

for every $x_o, x \in X$.

Suppose that X is a real linear topological space, f a finite convex function on X, and $x_o \in X$. Suppose that $f'(x_o; \cdot) = \phi$ belongs to X^*. Then

$$\phi(x) = \lim_{t \to 0} \frac{f(x_o+tx) - f(x_o)}{t}, \tag{1.4.2}$$

that is, the two sided limit exists for all $x \in X$. The functional ϕ is called the gradient of f at x_o, and is written $\nabla f(x_o)$.

1.5 Subgradients . [38] We now turn to a satisfactory generalization of the notion of gradient in the case of convex functions which are not differentiable. The appropriate concept is a "subgradient" of a convex function, the theory of which has been extensively developed in the last few years by Brøndsted [12] , Rockafellar [7] , and others.

Definition 1.5.1 . Let X be a real linear topological space and $f \in conv(X)$. Any $\phi \in X^*$ satisfying the inequality

$$f(y) \geq f(x_o) + \phi(y-x_o), \text{ for all } y \in X \tag{1.5.1}$$

is called a subgradient of f at x_o. The set of all such ϕ is the subdifferential of f at x_o, denoted by $\partial f(x_o)$.

Remark 1.5.2 . (a) If $\nabla f(x_o)$ exists, then it obviously belongs to $\partial f(x_o)$, and it can be seen that there is no other subgradient at x_o .

(b) If $x_o \notin \text{dom}(f)$, then $\partial f(x_o)$ is void by definition. On the other hand, if $x_o \in \text{dom}(f)$, then $\phi \in X^*$ is a subgradient of f at x_o if and only if $\phi(x) \leq f'(x_o;x)$ for all $x \in X$.

Example 1.5.3 . A norm is a proper convex function. If ϕ is a subgradient of the norm at x_o, then by (1.5.1)

$$||y|| \geq ||x_o|| + \phi(y-x_o) \quad \text{for all} \quad y \in X,$$

i.e. (i) $\phi(x_o) \geq ||x_o||$, and

(ii) $\phi(z) \leq ||z||$ for all $z \in X$ (put $z = y - x_o$).

Thus if $x_o \neq \theta$, then $\phi(x_o) = ||x_o||$ and $||\phi||=1$, and if $x_o = \theta$, then $\phi(y) \leq ||y||$, for all $y \in X$ implies that the unit ball $U(X^*)$ of the conjugate space is the subdifferential of the norm at θ .

Conversely, if $||\phi|| = 1$, $\phi(x) = ||x|| \neq 0$, then for any $y \in X$, we have

$$||y|| = ||x|| - \phi(x)+||y|| = ||x||+\phi(y-x)-\phi(y)+||y||$$

$$\geq ||x|| + \phi(y-x) \quad (\text{Since } ||y|| \geq \phi(y))$$

and hence ϕ is a subgradient of the norm at x.

<u>Theorem</u> 1.5.4. Let X be a locally convex space and $f \in \mathrm{conv}(X)$. If f is continuous at $x_o \in \mathrm{dom}(f)$, then $\partial f(x_o)$ is a nonempty weak* compact convex subset of X^*.

<u>Theorem</u> 1.5.5. (Moreau, Pshenichnii). Let X be a real locally convex space and $f \in \mathrm{conv}(X)$. Assume that f is continuous at x_o. Then for all $x \in X$,

$$f'(x_o;x) = \max\{\phi(x) : \phi \in \partial f(x_o)\}.$$

1.6 <u>Best Approximation</u>. Let M be a subset of a normed linear space X; then a point $x' \in M$ is said to be a <u>best approximation</u> (or <u>nearest point</u>) to $x \in X$ from M if $||x - x'|| = \inf\{||x-y|| : y \in M\}$. If each $x \in X$ has at least (resp. exactly) one best approximation in M, then M is called a <u>proximal</u> (resp. <u>Chebyshev</u>) subset of X. The map P_M which associates with each x the set of all x' (the best approximations of x) in M is called <u>best approximation operator</u> (or <u>metric projection</u>) supported by M. The scalar $\inf\{||x-y|| : y \in M\}$ is denoted by $d_M(x)$. It is easy to see that the function $d_M(x)$ satisfies a Lipschitz condition $d_M(x) - d_M(y) \le ||x-y||$, for all x and y in X.

The concept of a <u>sun</u> is originally due to Efimov and Stechkin [32]. However, various generalizations of it have been made by Vlasov [80] and we list them in the following: M is a set and P_M is its metric projection; M is called a

1) sun if $\forall x \notin M \ \exists\ y \in P_M(x)$ such that $y \in P_M(z)\ \forall\ z$ in the half ray $\overrightarrow{yx}$, issuing from y and passing through x;

2) α_o-sun if $\forall\ x \notin M\ \exists\ \ell_x : d_M(z) \geq ||z-x||\ \forall\ z \in \ell_x$, where ℓ_x is a half ray with vertex x;

3) α_1-sun if $\forall\ x \notin M, \forall\ \varepsilon > 0, \exists\ \ell_x : d_M(z) \geq ||z-x|| + d_M(x) - \varepsilon, \forall z \in \ell_x$;

4) α-sun if $\forall\ x \notin M\ \exists\ \ell_x : d_M(z) = ||z-x|| + d_M(x)\ \forall\ z \in \ell_x$;

5) β-sun if $\forall\ x \notin M, \forall\ R > 0\ \exists\ z : d_M(z) - d_M(x) = ||z-x|| = R$;

6) γ-sun if $\forall\ x \notin M, \forall R > 0\ \exists\ z_n : d_M(z_n) - d_M(x) \to R = ||z_n-x||$;

7) δ-sun if $\forall\ x \notin M\ \exists\ z_n \neq x : (d_M(z_n) - d_M(x))/||z_n-x|| \to 1$.

<u>Remark</u> 1.6.1. Valsov [80] has shown that 4) $\Longrightarrow$ 3) $\Longrightarrow$ 2) ; 1) $\Longrightarrow$ 4) $\Longrightarrow$ 5) $\Longrightarrow$ 6) $\Longrightarrow$ 7) and in a Banach space X , the class of γ-suns coincides with the class of δ-suns. It is also shown that in a space X with the property (M) (according to him such a space is said to belong to the class (CIUR)), every proximal γ-sun is an α-sun. It can be also easily checked that 3) $\Longrightarrow$ 6). This means that the concepts of $\alpha_1, \alpha, \beta, \gamma$, δ-suns in a Banach space with the property (M) are the same.

<u>Proposition</u> 1.6.2. [80] In a smooth space X, every α_1-sun is convex.

A set-valued metric projection P_M is said to be <u>upper semi-continuous</u> (<u>lower semi-continuous</u>), if the set $\{x \in X : P_M(x) \subset V\}$ is open (closed) for each open (closed) subset V of M.

A set $K \subset X$ is called <u>boundedly compact</u> if its intersection with any closed ball is compact. K is called <u>approximatively compact</u> if, for any $x \in X \sim K$ and $\{g_n\} \subset K$, the relation $||x - g_n|| \to d_K(x)$ implies that $\{g_n\}$ is compact in K.

It is known that [52, 71] every boundedly compact Chebyshev set and every approximatively compact Chebyshev set in a normed linear space X admit continuous metric projections.

1.7 <u>Brief Review</u>. One of the recent trends, in the theory of best approximation, is the study of convexity of Chebyshev sets. It is a well-known result that a Banach space X is reflexive and strictly convex if and only if every closed convex set in X is Chebyshev. There arises naturally the problem of characterizing the Banach spaces in which every Chebyshev set is convex. It is known that in a smooth Banach space X of finite dimension every Chebyshev set is convex. It is also known that [10] for every integer $n \geq 3$ there exists an n-dimensional non-smooth Banach space X with the property that every Chebyshev set in X is convex. However, it is not known whether the convexity of Chebyshev sets is true in Banach spaces of infinite dimension.

V. Klee [52] has shown that there is a close connection between the convexity of Chebyshev sets in Hilbert spaces and the following problem on farthest points: If K is a nonempty subset of a Hilbert space H such that each point of H admits a unique farthest point in K, then must K be a singleton ?

If the answer to this problem were in affirmative, then every Chebyshev set in H would be convex. There arises naturally the problem of characterizing the Banach spaces in which every nonempty set having unique farthest point property is a singleton. Partial answers to this problem have been provided by E. Asplund in [3] and V. Klee in [52]. It is proved in [3] that in R^n with an unsymmetric norm (i.e. positively homogeneous and subadditive), every nonempty set having unique farthest point property is a singleton. As for the infinite dimensional Banach spaces, it is shown that if the norm in X is the maximum of a countable family of linear functions, then nonempty sets in X having unique farthest point property are singletons. As a consequence, the spaces c_o and c are the two infinite dimensional classical Banach spaces having this property. Quite recently, J. Blatter [8] has shown that if K is a nonempty subset of a Banach space X such that the associated farthest point map is singlevalued and continuous in $\overline{co}\,(K)$, then K consists of a single point. In this context, he has introduced a concept of $\doteq$ compact set such that this set, when it has unique farthest point property, supports a continuous farthest point map. In this thesis, these sets are called M-compact sets, the prefix "M" symbolizing that all maximizing sequences in the set are considered. We study the M-compact sets in the setting of a space with the property (M). Spaces with the property (M) are generalization of locally uniformly

convex spaces introduced by A.R. Lovaglia [62] and have been studied in detail in Chapter 2. As has been shown by J. Blatter [8], in a Banach space X, every nonempty M-compact set having unique farthest point property is a singleton. This is a consequence of his main result which requires that the farthest point map be continuous. We show that in a normed linear space "admitting centres", every nonempty set having unique farthest point property is a singleton provided the farthest point map satisfies a mild continuity condition. We also prove an existence result on farthest points and thereby generalize the corresponding results of E. Asplund [2] and M. Edelstein [25].

In a non-Hilbert space X, no relationship has yet been established between the problem of convexity of Chebyshev sets and the problem of singletonness of sets having unique farthest point property, although many results concerning these problems run similar. Therefore, the problem of convexity of Chebyshev sets in an infinite dimensional Banach space has to be approached in a different way. In this direction, we note that so far only a partial answer has been obtained. The notions of bounded compactness and sun [32,78] have been introduced and L.P. Vlasov [77] has shown that in a normed linear space X every boundedly compact Chebyshev set is a sun. Further, if X is a smooth normed linear space, then every sun is convex [78]. N.V. Efimov and S.B. Stechkin [33] have generalized the concept of bounded

compactness and have introduced approximatively compact sets. It is proved that in a uniformly convex Banach space, every weakly sequentially closed set is approximatively compact. Characterizations of Banach spaces with this property have been given by I. Singer [71] . It has been shown by N.V. Efimov and S.B. Stechkin [33] that in a smooth and uniformly convex Banach space X, a Chebyshev set K is convex if and only if it is approximatively compact, in particular, when K is a weakly closed set in a Banach space X which is uniformly smooth in each direction and uniformly convex, this result has been obtained by V. Klee [52] .

Instead of prescribing additional conditions directly on the Chebyshev set K in order to be able to conclude that it is convex, one may impose, for the same purpose, conditions on the metric projection P_K. V. Klee [52] has proved that if K is a Chebyshev set in a smooth and reflexive space X and if every $x \in X \sim K$ admits a neighbourhood $V(x)$ such that $P_K|_{V(x)}$ is both strongly and weakly continuous, then K is convex. It has been shown by J. Blatter, P.D. Morris, and D.E. Wulbert [9] that the strong continuity of P_K can be dropped and the above result still holds with the lone assumption of weak continuity of P_K. On the other hand, L.P. Vlasov [80,81] has shown that if X^* is strictly convex and if P_K is strongly continuous, then K is convex. It may be noted that [71] every approximatively compact Chebyshev set supports a continuous metric projection. In Chapter 4, we prove

the converse of this result in a space with property (M), namely, the continuity of metric projection implies that the supporting set is approximatively compact. Moreover, it is shown that in such a space the domain of continuity of the metric projection supported by an arbitrary Chebyshev set is dense in the space.

It has been shown by V. Klee [52] that there is a connection between weak continuity of metric projections and weak closure of equidistant sets. He has considered properties P_1 and P_2 [52] to show that in a space X with the P_1-property (resp. P_2-property) every boundedly compact Chebyshev set supports a metric projection which is weakly continuous (resp. weakly continuous in a neighbourhood of each point). C.A. Kottman and Bor-Luh-Lin [54] have considered this problem with respect to both weak and bounded weak topologies. In generalizing Klee's result, they have shown that if X is smooth, $M = [\{x_\alpha\}_{\alpha \in A}]$ is a reflexive subspace, and for each α, $[x_\alpha]$ is a Chebyshev subspace and the equidistant set $E(-x_\alpha, x_\alpha)$ is bounded weakly closed, then the metric projection P_M is bounded weakly continuous. This gives rise to the problem of characterizing the spaces in which every equidistant set is weakly or bounded weakly closed. We prove that, in an ℓ^p-space $(1 < p < \infty)$, every equidistant set is bounded weakly closed. However, the characterization of such spaces is still an open problem.

For a detailed account of the recent development in the theory of best approximations, see the Appendix in [71] .

CHAPTER - II

A GENERALIZATION OF LOCAL UNIFORM CONVEXITY OF THE NORM

2.1 Introduction. A real normed linear space X is said to have the property

(M) if whenever $||x|| = 1, ||g_n|| \leq 1$ and $||x + g_n|| \to 2$, then there is a subsequence of $\{g_n\}$ which converges.

This is a generalization of the following notion introduced by Lovaglia [62] : X is called locally uniformly convex

(LUC) if whenever $||x|| = 1$, $||g_n|| \leq 1$ and $||x + g_n|| \to 2$, then $||x - g_n|| \to 0$.

In the present thesis, LUC spaces and, more generally, spaces with the property (M) provide, in a way, a natural setting for considering certain questions regarding "best approximations" and "farthest points". The purpose of this chapter is to study the property (M) of a normed linear space and its relationship with other properties such as the Efimov Stechkin property considered by Singer [71] , the property (h) which together with strict convexity is the property (H) introduced by Ky Fan and Glicksberg [34] , and the following property: A normed linear space X is said to satisfy the property

(S) if for any sequence $\{\psi_n\} \subset X^*$ with $||\psi_n|| \leq 1$, and for any $x \in X$ with $||x|| = 1$, the relation $\psi_n(x) \to 1$ implies that the sequence $\{\psi_n\}$ has a convergent subsequence.

It can be easily seen that ([38] , Lemma, p. 147) the property (S) is a weakening of the Smulian's criterion of strong smoothness of the norm. It turns out that for reflexive spaces there is a complete duality between the Efimov Stechkin property and the property (S).

A normed linear space X is said to have the <u>Efimov Stechkin property</u> [71] if each weakly sequentially closed set is approximatively compact. Singer has given the following characterizations of the Efimov Stechkin property :

<u>Theorem</u> 2.1.1. [71] For a Banach space X the following conditions are equivalent :

1° X has the Efimov Stechkin property.

2° Every weakly closed set in X is approximatively compact.

3° Every closed convex set in X is approximatively compact.

4° Every closed hyperplane in X is approximatively compact.

5° X is reflexive and has the property (h) : whenever $\{x_n\} \subset X$, $x \in X$, $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, then $||x_n - x|| \to 0$.

If X is strictly convex and has the Efimov- Stechkin property, then it is called an E-space. Such spaces have been first introduced by Ky Fan and Glicksberg in [34] and have been characterized by them in several ways. A detailed study of such spaces can also be found in [38] .

A normed linear space X is called <u>strongly smooth</u> if its norm is Frechet differentiable at every nonzero element of X. Let us recall that the norm is Frechet differentiable at $x \neq \theta$ if and only if any sequence $\{\phi_n\} \subset U(X^*)$, for which $\phi_n(x) \to ||x||$, is convergent. Clearly, the property (S) is a generalization of this concept. This property is stronger than the following A-property introduced by Anselone [53] :

(A) For $x \in X$, let $x^c = \{x^* \in X^*: ||x^*|| = ||x||$, and $x^*(x) = ||x||^2\}$. Then for each totally bounded subset T of X, the restriction of c to T admits a selection with a totally bounded range; that is, there is a function $s: T \to X^*$ such that $s(t) \in t^c$ for all $t \in T$ and the set sT is totally bounded.

Remark 2.1.2. We note that the property (S) implies that every selection s has a totally bounded range. For, if there is a selection s whose range sT is not totally bounded, then there exists a sequence $\{x_n^*\}$ in sT such that it has no strong cluster point. Since T is totally bounded and each x_n^* corresponds to an x_n in T, we can assume without any loss of generality that $x_n \to x_o$. Then

$$x_n^*(x_n) = ||x_n||^2, ||x_n^*|| = ||x_n||.$$

Further we can assume $x_o \neq \theta$, since otherwise $\{x_n^*\}$ would converge strongly to θ. Now set $z_n^* = x_n^*/||x_n||$. Then

$$z_n^*(x_n) = ||x_n||, \ ||z_n^*|| = 1, \text{ and}$$

hence $\lim_{n \to \infty} z_n^* (x_n) = \lim_{n \to \infty} z_n^* (x_o) = ||x_o||$. If X has the property (S), then $\{z_n^*\}$ is compact. This means $\{x_n^*\}$ is compact, which is a contradiction.

The following two propositions hold since they hold for the A-property [53] :

Proposition 2.1.3. Every infinite dimensional Banach space can be renormed so as to lack the property (S).

Proposition 2.1.4. There is a renormed version of ℓ^2 which lacks the property (S) even though its unit sphere is everywhere Gateaux-smooth and is Frechet-smooth except at two points.

By a recent result of Troyanski [75] every reflexive space can be given an equivalent norm which is strongly smooth. Thus, in particular, every reflexive space can be renormed so as to have the property (S).

2.2 Spaces with the Property (M) . We begin by observing that all finite dimensional Banach spaces have the Efimov Stechkin property and the properties (M) and (S). All IUC spaces have the property (M) and all strongly smooth spaces have the property (S). Indeed, we have

Theorem 2.2.1. X is IUC if and only if X is strictly convex and has the property (M).

Proof . It is clear that if X is IUC, then it is strictly convex and has the property (M). To prove the converse, suppose that $||x|| = 1$, $||g_n|| \leq 1$ and $||x + g_n|| \to 2$. Let $\{g_{n_i}\}$ be a subsequence of $\{g_n\}$ converging to g. Because of $||x + g_n|| \leq ||x|| + ||g_n|| \leq 2$, it is clear that $||g|| = 1$. Thus $||x + g|| = 2 = ||x|| + ||g||$, and strict convexity then implies that $x = g$. This shows that any convergent subsequence of $\{g_n\}$ converges to x. Further, any subsequence of $\{g_n\}$ contains a convergent subsequence. Therefore, $||x - g_n|| \to 0$ and X is IUC.

The following two theorems are instrumental in proving many results in this thesis.

Theorem 2.2.2. Let X be a normed linear space with the property (M). Then for each nonzero element $x \in X$, every sequence $\{g_n\} \subset U(X)$, satisfying $||x - g_n|| \to 1 + ||x||$, has a convergent subsequence.

Proof : Let $x \neq \theta$ be any element in X, and let $\{g_n\} \subset U(X)$ be such that $||x - g_n|| \to 1 + ||x||$. Choose $\psi_n \in S(X^*)$ such that

$$\psi_n(x - g_n) = ||x - g_n||. \tag{2.2.1}$$

Let $\psi_o \in S(X^*)$ and $\psi_o(x) = ||x||$. From $|\psi_n(g_n)| \leq 1$, and $|\psi_n(x)| \leq ||x||$ it follows that a subsequence $\{\psi_{n_i}\}$ of $\{\psi_n\}$ can be chosen so that both the sequences $\{\psi_{n_i}(x)\}$ and

$\{\psi_{n_i}(g_{n_i})\}$ are convergent. But $\psi_{n_i}(x) = \psi_{n_i}(g_{n_i}) + ||x-g_{n_i}||$ and hence $\lim\limits_{i \to \infty} \psi_{n_i}(x) \geq -1 + \lim\limits_{i \to \infty} ||x - g_{n_i}|| = -1 + 1 + ||x|| = ||x||$. This, together with $\lim\limits_{i \to \infty} \psi_{n_i}(x) \leq ||x||$, implies that $\psi_{n_i}(x) \to ||x||$. Now set

$$z_n = \frac{x - g_n}{||x-g_n||} \text{ and } z = \frac{x}{||x||}. \tag{2.2.2}$$

Then

$$2 = \lim_{i \to \infty} \psi_{n_i}(z_{n_i}+z) \leq \liminf_{i \to \infty} ||z_{n_i}+z|| \leq \limsup_{i \to \infty} ||z_{n_i}+z|| \leq 2,$$

and hence $||z_{n_i} + z|| \to 2$. By the property (M) of X, there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_n\}$ which converges. It follows that the sequence $\{g_n\}$ has a convergent subsequence. This proves the result.

<u>Corollary</u> 2.2.3. Let X be locally uniformly convex, and let K_n be defined by

$$K_n = U(X) \sim \operatorname{int} B[x, 1 + ||x|| - \varepsilon_n], \text{ where } x \neq \theta,\ 0 < \varepsilon_n < 1+||x||$$

and $\varepsilon_n \to 0$. Then diam $(K_n) \to 0$ as $n \to \infty$.

<u>Proof</u> : It is enough to prove that any sequence $\{g_n\}$ with $g_n \in K_n$ converges to $-x/||x||$. It is clear that $-x/||x|| \in K_n$ for each n and $1 + ||x|| - \varepsilon_n \leq ||x - g_n|| \leq 1 + ||x||$ which shows that $\{g_n\}$ is a maximizing sequence in $U(X)$ for the element x. By the local uniform convexity of X, the sequence

$\{z_{n_i}\}$ in the previous theorem converges to z and hence

$x-g_{n_i} \to \frac{x}{||x||}$, $\lim_{i \to \infty} ||x - g_{n_i}|| = \frac{x}{||x||} + x$. This shows that

$$g_{n_i} \to - \frac{x}{||x||} . \tag{2.2.3}$$

Since any subsequence of $\{g_n\}$ has a subsequence satisfying (2.2.3), it is clear that $-x/||x||$ is the only strong cluster point of $\{g_n\}$; that is, in other words $g_n \to -x/||x||$. This proves the assertion.

<u>Theorem</u> 2.2.4. Let X be a normed linear space with the property (M), and let x be a nonzero element of $U(X)$. Suppose that $\{g_n\}$ is a sequence in X with $||g_n|| \geq 1$, $||x - g_n|| \to 1 - ||x||$. Then the sequence $\{g_n\}$ is compact in X.

<u>Proof</u> : In view of $||x-g_n|| \geq ||g_n|| - ||x||$, and the relation $||x - g_n|| \to 1- ||x||$, it is clear that $||g_n|| \to 1$. Choose $\psi_n \in S(X^*)$ such that

$$\psi_n(g_n) = ||g_n||. \tag{2.2.4}$$

Then $||g_n|| = \psi_n(g_n-x+x) \leq ||x-g_n|| + \psi_n(x) \leq ||x-g_n|| + ||x||$.

This shows that $\lim_{n \to \infty} \psi_n(x) = ||x||$. Thus setting $z_n = g_n/||g_n||$ and $z = x/||x||$, we obtain

$$2 = \lim_{n \to \infty} \psi_n(z_n+z) \leq \liminf_{n \to \infty} ||z_n+z|| \leq \limsup_{n \to \infty} ||z_n+z|| \leq 2,$$

whence $||z_n + z|| \to 2$. As X has the property (M), the sequence $\{z_n\}$ has a convergent subsequence. Clearly, this also establishes the compactness of the sequence $\{g_n\}$ and the proof is complete.

Corollary 2.2.5. Let X be locally uniformly convex, and let K_n be defined by $K_n = B[x, 1 - ||x|| + \varepsilon_n] \sim \text{int } U(X)$, where $x \neq \theta$, $\varepsilon_n > 0$ and $\varepsilon_n \to 0$. Then diam $(K_n) \to 0$ as $n \to \infty$.

Proof : It is enough to show that any sequence $\{g_n\}$ with $g_n \in K_n$ converges to $x/||x||$. Clearly, $x/||x|| \in K_n$ for each n, $||g_n|| - ||x|| \leq ||x - g_n|| \leq 1 - ||x|| + \varepsilon_n$, and $||g_n|| \geq 1$ showing that $||x - g_n|| \to 1 - ||x||$. From the previous theorem $||z_n + z|| \to 2$ and then because of the local uniform convexity of the norm $z_n \to z$. This shows that $g_n \to x/||x||$ thus completing the proof.

2.3 Relationships among Various Properties.

Theorem 2.3.1. Let X be a normed linear space with the property (M). Then X has the property (h).

Proof : Let $\{y_n\} \subset X$, $y \in X$ such that $y_n \rightharpoonup y$ and $||y_n|| \to ||y||$. If $||y|| = 0$, then the assertion is trivial. Otherwise, write $x_n = y_n/||y_n||$ and $x = y/||y||$. Then $||x_n|| = 1 = ||x||$ and $x_n \rightharpoonup x$. Let $\phi \in S(X^*)$ be such that $\phi(x) = 1$, then

$$2 = \lim_{n \to \infty} \phi(x_n + x) \leq \liminf_{n \to \infty} ||x_n + x|| \leq \limsup_{n \to \infty} ||x_n + x|| \leq 2,$$

and hence $\lim\limits_{n \to \infty} ||x_n + x|| = 2$. By the property (M) of X, $\{x_n\}$ has a convergent subsequence. Also any subsequence of $\{x_n\}$ has a convergent subsequence. This shows, in view of $x_n \rightharpoonup x$, that $x_n \to x$. Hence the theorem is proved.

<u>Theorem</u> 2.3.2. X has the Efimov Stechkin property if and only if X^* has the property (S).

<u>Proof</u> : Let X have the Efimov Stechkin property, that is, X is reflexive and has the property (h). It will be required to show that for any $\{x_n\} \subset U(X)$ and $\phi \in S(X^*)$, the relation $\phi(x_n) \to 1$ will imply that the sequence $\{x_n\}$ has a convergent subsequence. By the reflexivity of X, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to some element x, say. From

$$1 = \phi(x) = \lim_{i \to \infty} \phi(x_{n_i}) \leq \liminf_{i \to \infty} ||x_{n_i}|| \leq \limsup_{i \to \infty} ||x_{n_i}|| \leq 1,$$

and $1 = \phi(x) \leq ||x|| \leq \liminf\limits_{i \to \infty} ||x_{n_i}|| \leq 1$, it follows that $||x_{n_i}|| \to 1 = ||x||$ and then by the property (h) of X, $x_{n_i} \to x$. This proves that X^* has the property (S). Conversely, let X^* have the property (S). To prove that X is reflexive, it is sufficient, in view of Jame's characterization of reflexivity [43], to show that each $\psi \in S(X^*)$ attains its supremum at some point of the unit sphere $S(X)$. As $||\psi|| = \sup\limits_{x \in S(X)} |\psi(x)|$, there exists a sequence $\{x_n\} \subset S(X)$ such that $\psi(x_n) \to ||\psi|| = 1$. But the space X^* has the property (S) and hence $\{x_n\}$ is compact. Any cluster point of $\{x_n\}$ is a point where ψ attains it

supremum on $S(X)$. This proves the reflexivity of X. To prove that X has the property (h), we consider a sequence $\{x_n\} \subset X$ which converges weakly to x and satisfies $||x_n|| \to ||x||$. Without loss of generality, we can assume $||x_n|| = 1 = ||x||$. Now choose $\psi \in S(X^*)$ such that $\psi(x) = 1$. Then we shall have $\psi(x_n) \to \psi(x) = 1$ which, in view of the property (S) of X^*, will imply that $\{x_n\}$ is compact. Since $x_n \rightharpoonup x$, x is the only strong cluster point of the sequence $\{x_n\}$. Hence the proof is complete.

The following corollary exhibits, for reflexive spaces, the complete duality between the Efimov Stechkin property and the property (S).

<u>Corollary</u> 2.3.3. If X is reflexive, then X has the property (S) $\Longleftrightarrow$ X^* has the Efimov Stechkin property.

<u>Proof</u> : The result follows trivially from the fact $X = X^{**}$ and Theorem 2.3.2.

<u>Theorem</u> 2.3.4. If X^* has the property (M), then X has the property (S).

<u>Proof</u> : Let $x_o \in S(X)$, $\{\psi_n\} \subset U(X^*)$ and $\psi_n(x_o) \to ||x_o|| = 1$. By Hahn-Banach theorem, there exists $\psi_o \in S(X^*)$ such that $\psi_o(x_o) = 1$. Thus

$$2 = \lim_{n\to\infty} (\psi_n+\psi_o)(x_o) \leq \liminf_{n\to\infty} ||\psi_n+\psi_o|| \leq \limsup_{n\to\infty} ||\psi_n+\psi_o|| \leq 2,$$

and hence $||\psi_n+\psi_o|| \to 2$ as $n \to \infty$. As X^* has the property (M), there is a subsequence of $\{\psi_n\}$ which converges. This proves that X has the property (S).

If X^* has the property (S), then X may not have the property (M) (see Example 2.3.13). But if, in addition, X has the following property:

(WM) if whenever $||x_n|| = 1 = ||x_o||$, $f_o \in S(X^*)$, $f_o(x_o) = 1$ and $||x_n + x_o|| \to 2$, then there exists a subsequence $\{x_{n_i}\}$ such that $f_o(x_{n_i}) \to 1$,

then we do have

<u>Theorem</u> 2.3.5. If X has the property (WM) and X^* has the property (S), then X has the property (M).

<u>Proof</u> : Let $x_n, x \in S(X)$ and $||x_n + x|| \to 2$. There exists $f \in S(X^*)$ such that $f(x) = 1$, and then by the property (WM) of X, a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ can be extracted such that $f(x_{n_i}) \to 1$. This, in view of the property (S) of X^*, implies that $\{x_{n_i}\}$ has a convergent subsequence and the result follows.

<u>Corollary</u> 2.3.6. If X has the Efimov Stechkin property and the property (WM), then it has the property (M).

<u>Proof</u> : This follows immediately from Theorems 2.3.2 and 2.3.5

<u>Corollary</u> 2.3.7. If (i) X is reflexive,

(ii) X has the property (S), and

(iii) X^* has the property (WM),

then X^* has the property (M).

Proof. This follows immediately from Theorem 2.3.5.

Theorem 2.3.8. If both of X and X^* have the Efimov Stechkin property, then both have the property (M).

Proof : We only prove that X has the property (M), the proof for X^* being similar. Let $x_n, x_o \in S(X)$ and $||x_n + x_o|| \to 2$. Choose $\{f_n\} \subset S(X^*)$ such that $f_n(x_n + x_o) = ||x_n + x_o||$. Then for any $z \in X$,

$$||x_n+x_o|| = f_n(x_n+z) + f_n(x_o-z) \leq ||x_n+z|| + f_n(x_o-z)$$

$$\leq 1 + ||z|| + f_n(x_o-z)$$

or $$||z|| \geq ||x_n + x_o|| - 1 + f_n(z-x_o) \tag{2.3.1}$$

Let f_o be any weak cluster point of the sequence $\{f_n\}$; then from (2.3.1)

$$||z|| \geq ||x_o|| + f_o(z-x_o), \text{ for all } z \in X.$$

This implies that $f_o(x_o) = ||x_o||$ and $||f_o|| = 1$. As X^* has the property (h), f_o will be a strong cluster point of the sequence $\{f_n\}$. Let $\{f_{n_i}\}$ be a subsequence of $\{f_n\}$ such that $f_{n_i} \to f_o$. Then we have $\lim_{i \to \infty} f_o \left(\frac{x_{n_i} + x_o}{||x_{n_i} + x_o||} \right) = 1$, and because of the property (S) of X^*, the sequence $\left\{ \frac{x_{n_i} + x_o}{||x_{n_i} + x_o||} \right\}$ has a

convergent subsequence. Hence the sequence $\{x_n\}$ is compact and this completes the proof.

If, in addition, X and X* are strictly convex, then we have the following theorem [34] due to Ky Fan and Glicksberg:

Theorem 2.3.9. If a normed linear space X and its conjugate space X* are E-spaces, then X and X* are locally uniformly convex.

Remark 2.3.10. The Theorem 2.3.8 provides a sufficient condition for a reflexive Banach space X to have the property (WM), namely, that the space X should be strongly smooth. In other words, X* should be an E-space. The proof follows from the fact that $f_n \to f_o$ and $f_o(x_o) = 1 = \lim_{x \to \infty} f_n(x_n) = \lim_{n \to \infty} f_o(x_n)$, where f_o, f_n, x_o, x_n are as defined in Theorem 2.3.8. Such a space need not have the property (M).

Example 2.3.11. An example of a reflexive space which has the property (M), but is not LUC. Let $X = \ell^p$ $(1 < p < \infty)$, and x be an arbitrary element in X. Denote

$$x = (x_1, \ldots, x_n, \ldots),\ x' = (0, x_2, \ldots, x_n, \ldots), \text{ and } x'' = (x_1, 0, x_3, \ldots, x_n, \ldots).$$

Now define a new norm $|||\cdot|||$ in X by

$$|||x||| = \max\{||x'||_p, ||x''||_p\}.$$

From $\frac{1}{2}(||x'||_p + ||x''||_p) \leq |||x||| \leq ||x||_p$, it follows that $\frac{1}{2}||x||_p \leq |||x||| \leq ||x||_p$ and hence the norm $|||\cdot|||$ is equivalent

to the ℓ^p-norm. That $(X, |||\cdot|||)$ is not strictly convex follows from the fact that if we take $x = (1,1,0,0,\ldots)$ and $y = (-1,1,0,0,\ldots)$, then $|||x||| = 1 = |||y|||$ and $|||x + y||| = 2$, whereas $y \neq \lambda x$ for any $\lambda > 0$. Thus X cannot be IUC. To prove that X has the property (M), we take $x, g_n \in X$ with $|||x||| = 1$, $|||g_n||| \leq 1$ and $|||x + g_n||| \to 2$. By definition of the new norm, $|||x + g_n||| = \max \{||x'+g_n'||_p, ||x''+g_n''||_p\}$. This requires two cases to be considered :

(i) $||x'+g_n'||_p \geq ||x''+g_n''||_p$ for infinity of n, and

(ii) $||x'+g_n'||_p < ||x''+g_n''||_p$ for sufficiently large n.

Suppose that (i) holds. Since we are interested in extracting a convergent subsequence of $\{g_n\}$, we may assume without loss of generality that (i) holds for all sufficiently large n. Thus $||x'+g_n'||_p \to 2$, which shows that $||x'||_p = 1$, $||g_n'||_p \to 1$. The ℓ^p-norm being (locally) uniformly convex, $||x'-g_n'||_p \to 0$. As the sequence of real numbers $\{g_n(1)\}$ is bounded, it follows that the sequence $\{g_n\}$ is compact. Case (ii) is similar and hence the result follows.

<u>Example</u> 2.3.12. <u>An example of a nonreflexive space which has the property (M), but is not IUC.</u> Let $X = c_o$, and x, x', x'' be as in the previous example. Let $||\cdot||_1$ be the norm constructed by Day [70]. It is known that this norm is IUC. Now enumerate the components of x in the order of nonincreasing absolute values. Let $(x(\alpha_1), x(\alpha_2), \ldots, x(\alpha_n), \ldots)$ be such an enumeration, and let $x(1) = x_1$ and $x(2) = x_2$ fall at the n_o th and the m_o th position respectively. (Note that m_o and n_o may be infinite)
Then

$$||x||_1^2 = \left(\left|\frac{x(\alpha_1)}{2}\right|^2 + \ldots + \left|\frac{x(\alpha_{n_o-1})}{2^{n_o-1}}\right|^2\right) + \left|\frac{x(1)}{2^{n_o}}\right|^2 + \left(\left|\frac{x(\alpha_{n_o+1})}{2^{n_o+1}}\right|^2 + \ldots\right)$$

$$= \alpha + \left|\frac{x(1)}{2^{n_o}}\right|^2 + \beta \text{ , (say),} \tag{2.3.2}$$

and $$||x'||_1^2 = \alpha + \left(\left|\frac{x(\alpha_{n_o+1})}{2^{n_o}}\right|^2 + \ldots\right) = \alpha + 4\beta \tag{2.3.3}$$

If $m_o > n_o$, then

$$||x''||_1^2 = \alpha + \left|\frac{x(1)}{2^{n_o}}\right|^2 + \text{ some positive quantity ,} \tag{2.3.4}$$

and if $m_o < n_o$, then

$$||x''||_1^2 = \text{some positive quantity} + \left|\frac{x(1)}{2^{n_o-1}}\right|^2 + 4\beta. \tag{2.3.5}$$

Thus $$||x||_1^2 = \alpha + \beta + \left|\frac{x(1)}{2^{n_o}}\right|^2 \geq \alpha + \beta \geq \frac{\alpha}{4} + \beta = \frac{1}{4}\, ||x'||_1^2 \text{ ,} \tag{2.3.6}$$

and similarly $$||x||_1^2 \geq \frac{1}{4}\, ||x''||_1^2 . \tag{2.3.7}$$

Relations (2.3.6) and (2.3.7) together imply that

$$\max \quad \{||x'||_1 , ||x''||_1\} \leq 2\, ||x||_1. \tag{2.3.8}$$

$$\text{Moreover}\quad ||x'||_1^2 + ||x''||_1^2 = \begin{cases} \alpha+4\beta+\alpha+\left|\dfrac{x(1)}{2^{n_o}}\right|^2 + \text{some positive number if } m_o > n_o, \\ \\ \alpha+4\beta+4\beta+\left|\dfrac{x(1)}{2^{n_o-1}}\right|^2 + \text{some positive number if } m_o < n_o. \end{cases}$$

$$\geq \begin{cases} \alpha+\beta+\left|\dfrac{x(1)}{2^{n_o}}\right|^2 & \text{if } m_o > n_o \\ \alpha+\beta+4\left|\dfrac{x(1)}{2^{n_o}}\right|^2 & \text{if } m_o < n_o, \end{cases}$$

and hence $||x'||_1^2 + ||x''||_1^2 \geq \alpha+\beta+\left|\dfrac{x(1)}{2^{n_o}}\right|^2 = ||x||_1^2$. Thus, if we define a new norm $|||\cdot|||$ by

$$|||x||| = \max\{||x'||_1, ||x''||_1\},$$

then $\frac{1}{4}(||x'||_1^2 + ||x''||_1^2) \leq \frac{1}{4}(||x'||_1+||x''||_1)^2 \leq |||x|||^2 \leq 4||x||_1^2$,

i.e., $\frac{1}{2}||x||_1 \leq |||x||| \leq 2\,||x||_1$,

which shows that the two norms $||\cdot||_1$ and $|||\cdot|||$ are equivalent. The proof that $(X,|||\cdot|||)$ has the property (M) and is not IUC is the same as in the previous example.

<u>Example</u> 2.3.12. <u>An example of a space which has the Efimov Stechkin property but lacks the property</u> (M). Let $X = \ell^p$ $(1 < p < \infty)$, and x be any arbitrary element in X. Define a new norm in X by

$$|||x||| = |x_1| + ||x'||_p,$$

where $x = (x_1, x_2, \ldots, x_n, \ldots)$ and $x' = (0, x_2, x_3, \ldots, x_4, \ldots)$. Clearly $||x||_p \leq |||x||| \leq 2||x||_p$ and hence the ℓ^p-norm is equivalent to $|||\cdot|||$. To prove that X has the Efimov Stechkin property, we need only show that if satisfies the property (h). Let $x_n, x_o \in X$, $x_n \rightharpoonup x_o$, and $|||x_n||| \to |||x_o|||$. As weak convergence in ℓ^p $(1 < p < \infty)$ implies pointwise convergence, this means that $x_n(1) \to x_o(1)$ and $||x_n'||_p \to ||x_o'||_p$. But the truncated sequence $\{x_n'\}$ is weakly convergent to x_o' since it is bounded and pointwise convergent. Now, because of the uniform convexity of the ℓ^p-norm, $||x_n' - x_o'||_p \to 0$ and hence $|||x_n - x_o||| \to 0$, showing that X has the property (h). To show that X does not have the property (M), we take $x = e_1$, $g_n = e_n$, where $e_n(m) = \delta_{nm}$, the Kronecker delta. Then $|||x||| = 1 = |||g_n|||$, $|||x + g_n||| = 2$, but the sequence $\{g_n\}$ has no convergent subsequence.

We note that the norm $|||\cdot|||$ is not strictly convex and hence its dual norm is not smooth. As X has the Efimov Stechkin property, the dual norm has the property (S).

CHAPTER - III

ON FARTHEST POINTS OF SETS

3.1 INTRODUCTION. One of the most interesting and hitherto unsolved problems in the approximation theory is the following:

Problem 3.1.1. If every point of a normed linear space X admits a unique farthest point in a given bounded set K, then must K be a singleton?

This problem has been discussed by Klee [52] in connection with the convexity problem of a Chebyshev set in a Hilbert space. Partial answers to this problem have been provided by Asplund [3], Blatter [8], and Klee [52]. Klee [52] has considered the particular cases such as (i) K is a compact subset of a Banach space, (ii) K is a totally bounded subset of a strictly convex Banach space, and (iii) X is finite dimensional with a polyhedral unit cell. In each of these cases, he has shown that the solution to the above problem is in the affirmative. Asplund [3] has extended all these results of Klee in both finite and infinite dimensional situations. To this end, he has considered the following cases :

(a) X is a finite dimensional Banach space with an unsymmetric norm.

(b) The norm in X is given by the maximum of a countable family of linear functions. In each of these cases, every nonempty set K having unique farthest point property is a singleton. It may be noted that the spaces in (b) above are the only infinite dimensional Banach spaces, known till date, in which every nonempty set having unique farthest point property is a singleton. The latest development in this direction is possibly due to Blatter [8]. He has shown that if the farthest point map associated with a set K in a Banach space X is continuous, then K must consist of a single point.

The question of existence of farthest points in an arbitrary bounded and closed set has been treated by Asplund [2] and Edelstein [25]. Edelstein [25] has shown that in a uniformly convex Banach space X, every closed and bounded set K admits farthest points from points of a dense subset of X. Asplund [2] has extended this result replacing the uniform convexity assumption by reflexivity and local uniform convexity of the norm.

The object of this chapter is to continue the study of existence and uniqueness of farthest points in a set. To this end, the notion of an M-compact set has been introduced and studied in detail. It turns out that such a set, when it has unique farthest point property, supports a continuous farthest point map. The continuity behavior of the farthest point map

supported by an arbitrary set having unique farthest point property is also investigated. Finally, it is shown that in a normed linear space "admitting centres", every nonempty set having unique farthest point property is a singleton provided the farthest point map satisfies a mild continuity condition.

3.2 <u>Existence of Farthest Points.</u> Let K be a bounded subset of a normed linear space X. We shall denote by $F_K(x)$ the farthest distance of $x \in X$ from elements in K; that is, $F_K(x) = \sup \{||x - y|| : y \in K\}$. The set of points in K at which $F_K(x)$ is attained will be denoted by $q(x)$. This relationship between x in X and the set of points $q(x)$ in K gives rise to a map $q:X \to K$ which we shall call the <u>farthest point map</u> (or the <u>anti-projection operator</u>) supported by K. Every element of $q(x)$ will be called a <u>farthest point</u> in K. The collection of all farthest points in K will be denoted by far(K). If $q(x)$ contains at least (exactly) one element for each $x \in X$, then K is said to have <u>farthest point property</u> (<u>unique farthest point property</u>).

<u>Proposition</u> 3.2.1. The function F_K is a convex function satisfying a Lipschitz condition $|F_K(x) - F_K(y)| \leq ||x-y||$, for all $x,y \in X$.

<u>Proof</u> : The convexity of the function F_K follows from the following inequalities:

$$F_K(\lambda x+(1-\lambda)y)=\sup_{z\varepsilon K} ||\lambda x+(1-\lambda)y-z|| \leq \lambda \sup_{z\varepsilon K} ||x-z|| + (1-\lambda) \sup_{z\varepsilon K} ||y-z||$$

$$= \lambda F_K(x) + (1-\lambda) F_K(y), \text{ for all } \lambda \text{ with } 0 \leq \lambda \leq 1.$$

To prove the Lipschitz property of F_K, take any element z in K. Then $||x-z|| \leq ||x-y|| + ||y-z||$ and hence $F_K(x) \leq ||x-y|| + F_K(y)$. This proves the result.

Lemma 3.2.2. (Asplund [2]) Let K be a bounded subset of a Banach space X. Then there exists a dense subset E, which is the intersection of a countable family of open and dense subsets of X, with the property that every element of $\partial F_K(x)$, for x in E, is of unit norm.

A slightly weaker form of the following theorem has been proved by Asplund [2], namely, when K is a bounded and closed subset of a reflexive, locally uniformly convex Banach space.

Theorem 3.2.3. Let K be either (a) a bounded, closed subset of a reflexive Banach space X with the property (M), or (b) a bounded, weakly sequentially closed subset of a reflexive Banach space X with the property (WM). Then the set D of all points in X that have farthest points in K is dense in X. Moreover, $X \sim D$ is of the first category in X.

Proof : First, let K be a bounded, closed subset of a reflexive Banach space X satisfying the property (M). We shall prove that

the set D, as described in the theorem, contains the set E, where E is as in Lemma 3.2.2. This will prove the result because of the nature of E.

Let y be an arbitrary element of E. Without loss of generality, we may assume that $y = \theta$, and $F_K(\theta) = 1$. Let $x^* \in \partial F_K(\theta)$. Since X is reflexive, x^* will attain its infimum on the unit sphere $S(X)$ at some point x, say. Then from the definition of a subgradient, we have

$$F_K(-x) \geq F_K(\theta) + <-x-\theta, x^*>,$$

and hence

$$1 = <-x, x^*> \leq F_K(-x) - F_K(\theta) \leq ||-x-\theta|| = 1.$$

This shows that $F_K(-x) = F_K(\theta) + 1 = 2$ and hence the ball $B[-x,2]$ is the smallest ball with centre $-x$ that contains K. So we may find $z_n \in K$ such that $2 - \frac{1}{n} \leq ||-x-z_n|| \leq 2$; that is, $\lim\limits_{n \to \infty} ||x + z_n|| = 2$, where $||x|| = 1$ and $||z_n|| \leq 1$. Since X has the property (M), this shows that the sequence $\{z_n\}$ has a convergent subsequence. The set K being closed, any strong cluster point of the sequence $\{z_n\}$ is in K. But, in view of $\lim\limits_{n \to \infty} ||x + z_n|| = 2$, such a cluster point is of unit norm. As $F_K(\theta) = 1$, it follows that all the cluster points of the sequence $\{z_n\}$ are farthest to θ. This proves the theorem.

Now, in order to prove (b), we observe from (a) that because of the relation $2 - \frac{1}{n} \leq ||x+z_n|| \leq ||x|| + ||z_n|| \leq 2$ we may assume $||z_n|| = 1, ||x|| = 1$ and $||x + z_n|| \to 2$. Since $\langle -x,x^* \rangle = 1$ and X has the property (WM), it follows that $\langle -z_{n_i}, x^* \rangle \to 1$ for some subsequence $\{z_{n_i}\}$ of $\{z_n\}$. If z_o is any weak cluster point of the sequence $\{z_{n_i}\}$, then $\langle -z_o, x^* \rangle = 1$. This together with $||z_o|| \leq \liminf_{j \to \infty} ||z_{n_{i_j}}|| = 1$, where $z_{n_{i_j}} \rightharpoonup z_o$, implies that $||z_o|| = 1$. But $\{z_n\} \subset K$ and K is weakly sequentially closed. Hence $z_o \in K$ and it is farthest to θ. This proves the theorem.

Following Mazur [63] , we shall say that a normed linear space X has the property (I) if every closed and bounded convex set in X can be represented as the intersection of a family of closed balls. Edelstein [25] has proved that in a uniformly convex Banach space X with the property (I) , the relation $\overline{co}(K) = \overline{co}(\text{far}(K))$ holds for every bounded and closed set K in X . The following is a generalization of this result.

Theorem 3.2.4. Let K and X be as in Theorem 3.2.3. In addition, let X have the property (I). Then $\overline{co}(K) = \overline{co}(\text{far}(K))$.

Proof : Since far $(K) \subset K$, it follows that $\overline{co}(\text{far}(K)) \subset \overline{co}(K)$. To prove the reverse inclusion, suppose $x \notin \overline{co}(\text{far}(K))$. Then, by the property (I), there exists a closed ball

$$B[c_o,r] = \{y \in X : ||y-c_o|| \leq r\} \quad ,$$

where $c_o \in x$ and $r > 0$, such that $\overline{co}(\mathrm{far}(K)) \subset B[c_o,r]$ but $x \notin B[c_o,r]$. A number α can now be chosen such that $0 < \alpha < 1$ and $\alpha||x-c_o|| > r$. By Theorem 3.2.3, there is an element $c \in X$ admitting a farthest point in K and satisfying the inequality

$$||c-c_o|| < \alpha||x-c_o|| - r. \tag{3.2.1}$$

If $s \in K$ is a farthest point from c, that is, if $s \in \mathrm{far}(K)$, then

$$||s-c|| \leq ||s-c_o|| + ||c_o-c|| < ||s-c_o|| + \alpha||x-c_o|| - r$$

$$\leq \alpha||x-c_o|| \text{ (since } s \in \mathrm{far}(K) \subset \overline{co}(\mathrm{far}(K)) \subset B[c_o,r])$$

which implies that $F_K(c) < \alpha||x-c_o||$. But F_K is continuous and c can be taken arbitrarily close to c_o. Thus $F_K(c_o) \leq \alpha||x-c_o||$ and so

$$||c_o-z|| \leq \alpha||x-c_o|| \quad , \quad \text{for all } z \in K,$$

that is, $K \subset B[c_o, \alpha||x-c_o||]$ and hence $\overline{co}(K) \subset B[c_o,\alpha||x-c_o||]$. But $x \notin B[c_o, \alpha||x-c_o||]$ since $0 < \alpha < 1$, which shows that $x \notin \overline{co}(K)$. Thus $\overline{co}(K) \subset \overline{co}(\mathrm{far}(K))$ and the proof is complete.

3.3 M. Compact Sets. Consider a sequence $\{g_n\} \subset K$ and an $x \in X$ such that $||x-g_n|| \to F_K(x)$. Such a sequence $\{g_n\}$ will be called a maximizing sequence for x.

Definition 3.3.1. A set $K \subset X$ will be called M-compact (or $\hat{}$ compact, see [8]) if every maximizing sequence in K for any $x \in X$ is compact in K.

Clearly, every compact set is M-compact. To show that the converse is not necessarily true, we observe from Theorem 2.2.2 that for any $x \neq \theta$ in X and any sequence $\{g_n\} \subset U(X)$ with $||x-g_n|| \to 1 + ||x||$, the sequence $\{g_n\}$ is compact in $U(X)$. Thus every maximizing sequence in $U(X)$ for any point x, other than the centre of the ball, is compact in $U(X)$. However, this does not imply that the ball $U(X)$ is M-compact, since the compactness of the maximizing sequences for $x = \theta$ has not yet been resolved. It can be easily seen that unless the unit sphere is compact, the maximizing sequences for $x = \theta$ may not be compact. Now consider a point $x_0 \notin U(X)$ and the set $K = \{x_0\} \cup U(X)$. Since every maximizing sequence in K contains either $\{x_0\}$ as a subsequence or a subsequence in $U(X)$, it is clear that K is M-compact. A still more general type of M-compact sets can be obtained from the following proposition.

Proposition 3.3.2. Let X be a reflexive Banach space having the property (M), and let $K \subset X$ be a compact set such that

it intersects the complement of the ball $U(X)$. Then both $U(X) \cup K$ and $\overline{co}\,(U(X) \cup K)$ are M-compact.

<u>Proof</u> : Since every maximizing sequence in $U(X) \cup K$ has a subsequence either in $U(X)$ or in K, Theorem 2.2.2 together with the compactness of K implies that $U(X) \cup K$ is M-compact. In order to prove that $\overline{co}\,(U(X) \cup K)$ is M-compact, it is sufficient, in view of

$$\overline{co}\,(U(X) \cup K) = \overline{co}\,(U(X) \cup \overline{co}\,K) = co\,(U(X) \cup \overline{co}\,K)$$

(by Lemma 1.3.1), to consider the case for the set $co(U(X) \cup K)$, where K is compact convex.

Let $\{g_n\} \subset co(U(X) \cup K)$ be a maximizing sequence for an element x in X. Then g_n can be written in the form

$$g_n = \lambda_n z_n + (1-\lambda_n) k_n, \; 0 \leq \lambda_n \leq 1, \; z_n \in U(X), \text{ and } k_n \in K.$$

We can choose subsequences such that $\lambda_{n_i} \to \lambda_o$ and $k_{n_i} \to k_o$. Let $||x-g_n|| \to d$, say. Then

$$\begin{aligned} d &\leq \lambda_o \liminf_{i \to \infty} ||x-z_{n_i}|| + (1-\lambda_o) ||x-k_o|| \\ &\leq \lambda_o \limsup_{i \to \infty} ||x-z_{n_i}|| + (1-\lambda_o) ||x-k_o|| \\ &\leq d. \end{aligned}$$

We have now three cases to consider. First, if $0 < \lambda_o < 1$, then $||x-z_{n_i}|| \to d$, $||x-k_o||=d$, and $d = 1 + ||x||$. Clearly, in this case $x \neq \theta$. The sequence $\{z_{n_i}\}$ being a maximizing

sequence in $U(X)$ is compact. This shows that the sequence $\{g_n\}$ is compact. In the case when $\lambda_o = 0$, k_o is a strong cluster point of $\{g_n\}$. Finally, when $\lambda_o = 1$, $\{z_{n_i}\}$ is a maximizing sequence in $U(X)$ for x and hence is compact. Thus the sequence $\{g_n\}$ is also compact. This shows that in each case $\{g_n\}$ is compact. This proves the result.

We shall now study some properties of M-compact sets.

Proposition 3.3.3. Every M-compact set has farthest point property.

Proof: This follows immediately from the definition of an M-compact set.

Proposition 3.3.4. The closure of an M-compact set is M-compact

Proof : Let K be an M-compact set in a normed linear space X, and let $\{g_n\} \subset \bar{K}$, $x \in X$, and $||x-g_n|| \to F_{\bar{K}}(x)$. Since $K \subset B[x, F_K(x)]$ implies $\bar{K} \subset B[x, F_K(x)]$, $F_K(x) = F_{\bar{K}}(x)$. For each n, an element $g_n' \in K$ can be so chosen that $||g_n-g_n'|| < \frac{1}{n}$, and since

$$\big|\, ||x-g_n|| - ||x-g_n'|| \,\big| \leq ||g_n-g_n'|| , \tag{3.3.1}$$

the sequence $\{g_n'\}$ is a maximizing sequence in K for the element x. As K is M-compact, this shows that the sequence $\{g_n'\}$ is compact in K. However, the sequences $\{g_n\}$ and $\{g_n'\}$ have the same set of cluster points. Hence $\{g_n\}$ is compact in $\bar{K}$ and this completes the proof.

Example 3.3.5. We give an example of an M-compact set which is not closed. Let K be an open square together with its four extreme points in the two diemnsional Euclidean space E^2. Since the farthest points in K are the four extreme points, K is M-compact. However, K is not a closed set.

Proposition 3.3.6. Let X be a normed linear space having the property (M), and let K be an M-compact set consisting of more than one element. Then the set $K_a = K + aU(X)$, $a > 0$, is also M-compact.

Proof. Let $x \varepsilon X$ and y be any element in K_a. Then y can be written in the form $y = u + av$, where $u \varepsilon K$ and $v \varepsilon U(X)$. In computing the value of $F_{K_a}(x)$ we see that

$$||x-y|| \leq ||x-u|| + a\,||v|| \leq F_K(x) + a,$$

and hence $F_{K_a}(x) \leq F_K(x) + a$. If $z \varepsilon K$ is farthest to x (note that K has farthest point property), then $z+a\frac{z-x}{||z-x||} \varepsilon K_a$ and

$$||x-z-a\frac{z-x}{||z-x||}|| = ||x-z|| + a = F_K(x) + a.$$

Hence $F_{K_a}(x) = F_K(x) + a$. Now, let $\{g_n\} \subset K_a$ be a maximizing sequence for an element $x \varepsilon X$; i.e.,

$$||x-g_n|| \to F_K(x) + a \qquad (3.3.2)$$

Expressing g_n in the form $g_n = u_n + a\,v_n$, we see that

$$F_K(x) + a = \lim_{n \to \infty} ||x-g_n|| \leq \liminf_{n \to \infty} ||x-u_n|| + a$$

$$\leq \limsup_{n \to \infty} ||x-u_n|| + a \leq F_K(x) + a.$$

This shows that $||x-u_n|| \to F_K(x)$. But then, by the M-compactness of K, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \to u_o$, $u_o \in K$, Therefore,

$$\lim_{n \to \infty} ||x-u_{n_i} - a\, v_{n_i}|| = \lim_{i \to \infty} ||(x-u_o) - a\, v_{n_i}|| = ||x-u_o|| + a;$$

that is,

$$\lim_{i \to \infty} ||\frac{1}{a}(x-u_o) - v_{n_i}|| = 1 + \frac{1}{a} ||x-u_o||. \qquad (3.3.3)$$

As K consists of more than one element, $||x-u_o|| = F_K(x) \neq 0$, and hence, applying Theorem 2.2.2, we see that $\{v_{n_i}\}$ is compact in $U(X)$. This shows that the sequence $\{g_n\}$ is compact in K_a. Hence the proof is complete.

Proposition 3.3.7. [8] The farthest point map supported by an M-compact set is upper semi-continuous.

Proof : Let K be an M-compact set and $q:X \to K$ be the associated farthest point map. Let x be any point of X. We will show that q is upper semi-continuous (u.s.c.) at x. Consider any open set W containing the set $q(x)$. Let

$$M = \{z \in X : q(z) \subset W\}.$$

Our assertion will be valid if we show that M is an open set. To this end we take a sequence $\{z_n\}$ in X such that $z_n \to z_o$, where z_o is any element of M, and show that $\{z_n\}$ is eventually in M. Suppose that it is not true; then there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $q(z_{n_i}) \cap W^c \neq \phi$, where W^c denotes the complement of W in X. let $y_{n_i} \in q(z_{n_i}) \cap W^c$, then

$$||z_{n_i} - y_{n_i}|| = F_K(z_{n_i}),$$

and since $|\,||z_{n_i} - y_{n_i}|| - ||z_o - y_{n_i}||\,| \leq ||z_{n_i} - z_o||$, we see that $||z_o - y_{n_i}|| \to F_K(z_o)$. Thus $\{y_{n_i}\}$ is a maximizing sequence in K for the element z_o and, by the M-compactness of the set K, it has a convergent subsequence. Suppose that y_o is a cluster point of the sequence $\{y_{n_i}\}$. Then $y_o \in q(z_o)$ and since W^c is closed, y_o is also in W^c. This shows that $y_o \in q(z_o) \cap W^c$ and hence $z_o \notin M$. This contradicts the fact that $z_o \in M$ and hence the proof is complete.

<u>Example</u> 3.3.8. We give an example of a set which supports a u.s.c. farthest point map, but is not M-compact. To this end consider the unit ball $U(X)$ of a normed linear space X which satisfies the property (M). Clearly, $U(X)$ is not M-compact, but we will show that the farthest point map $q: X \to U(X)$ is u.s.c.

Let $x \in X$ and W be an open set containing $q(x)$. Denote

$$M = \{z \in X : q(z) \subset W\}.$$

It is required to show that M is open. If $\theta \varepsilon M$, then because of $q(\theta) = S(X)$, $M = X$ and hence M is open. If $\theta \notin M$, then the proof is similar to that of Proposition 3.3.7, since every maximizing sequence in $U(X)$ for an element $x \neq \theta$ is compact in $U(X)$. This completes the proof.

<u>Definition</u> 3.3.9. A centre (or Chebyshev centre) of a bounded, nonempty set K in a normed linear space X is an element $x_o \varepsilon X$ for which $F_K(x_o) = \inf_{x \varepsilon X} \sup_{y \varepsilon K} ||x-y||$. The number $F_K(x_o)$ is called the Chebysher radius of K and is denoted by $r(K)$.

Clearly, $r(K)$ is the radius of the smallest ball in X (if one exists) which contains the set K. The collection of the centres of all such balls is denoted by $E(K)$. A n.a.s.c. for x_o to belong to $E(K)$ is that $\theta \varepsilon \partial F_K(x_o)$. However, the determination of such an element x_o always depends on our ability to subdifferentiate the function F_K.

In the following theorem, a formula for the subdifferential of the function F_K, where K is an M-compact set, at a point x in X is obtained. Such a formula for the subdifferential at a point of a function f equal to the supremum of a family of convex functions f_α, α ranging over a compact set Ω, has been given by Valadier [76]. The same for F_K, where K is a compact set, has been observed by Holmes [38,p. 182].

The first part of the proof of the following theorem is more or less the same as in [38,p. 179] .

Theorem 3.3.10. Let X be a normed linear space, K an M-compact set and $x_o \in X$. Let $f_y(x) = ||x-y||$. Then

$$\partial F_K(x_o) = \text{weak*-closed convex hull of} \bigcup \{\partial f_y(x_o) : y \in q(x_o)\}$$
$$= \text{weak* closed convex hull of} \bigcup \{\psi \in S(X^*) : \psi(x_o-y) = F_K(x_o), \text{ for some } y \in K\} \tag{3.3.4}$$

Proof : The function $F_K(x)$ is Lipschitzian and is a proper convex function on X. Hence, by Theorem 1.5.4, it is sub-differentiable at every point of X. Forming difference quotients, we see that

$$\frac{F_K(x_o+tx)-F_K(x_o)}{t} \geq \frac{||x_o+ tx-y|| - ||x_o-y||}{t}, \text{for } y \in q(x_o) \text{ and } t > 0.$$

Since both of F_K and the norm are convex functions, each of them has a right directional derivative (by Theorem 1.4.1) and hence on passing to the limit $t \to +0$

$$F'_K(x_o;x) \geq f'_y(x_o;x) , \text{ for all } y \in q(x_o), x \in X. \tag{3.3.5}$$

From Remark 1.5.2 (b) it follows that every subgradient of f_y at x_o is also a subgradient of F_K at x_o and hence

$$\partial F_K(x_o) \supset \partial f_y(x_o), \text{ for all } y \in q(x_o) \tag{3.3.6}$$

and therefore, the inclusion from right to left in (3.3.4) is valid.

In order to prove the reverse inclusion, it is sufficient, in view of Strong Separation Theorem (Theorem 1.3.3), to show that any weak* closed half-space containing $\cup\{\partial f_y(x_o) : y \in q(x_o)\}$ (and hence its weak* closed convex hull) must also contain $\partial F_K(x_o)$. That is, if for some $z \in X$ we have

$$\langle z, \psi \rangle \leq \lambda \quad \forall\, \psi \in \partial f_y(x_o) \text{ and } \forall\, y \in q(x_o),$$

then the same inequality also holds for every $\psi \in \partial F_K(x_o)$. Using Moreau-Pshenichnii formula (Theorem 1.5.5), this amounts to showing that

$$\sup_{y \in q(x_o)} f'_y(x_o; z) \leq \lambda$$

implies $\quad F'_K(x_o; z) \leq \lambda$.

Thus it will suffice to show that for any fixed $z \in X$, there is an index $y_o \in q(x_o)$ such that

$$F'_K(x_o; z) \leq f'_{y_o}(x_o; z) \tag{3.3.7}$$

The relations (3.3.5) and (3.3.7) together will imply that

$$F'_K(x_o; z) = \max\{f'_y(x_o; z) : y \in q(x_o)\} \tag{3.3.8}$$

Any sequence $\{g_n\}$ with $g_n \in q(x_o + \frac{1}{n} z)$ is a maximizing sequence in K for the point x_o and, by the M-compactness of K, contains a subsequence converging to some element y_o of $q(x_o)$. Without any loss of generality, we may assume that $g_n \to y_o$. Then by Theorem 1.4.1 and Remark 1.4.2,

$$F'_K(x_o;z) \leq \frac{F_K(x_o + \frac{1}{n} z) - F_K(x_o)}{\frac{1}{n}} = \frac{||x_o + \frac{1}{n} z - g_n|| - ||x_o - y_o||}{\frac{1}{n}}$$

$$\leq \frac{||x_o + \frac{1}{n} z - g_n|| - ||x_o - g_n||}{\frac{1}{n}}, \text{ for } n \geq 1. \qquad (3.3.9)$$

If ϕ_n is a subgradient of the norm at $x_o + \frac{1}{n} z - g_n$, then

$$||x_o - g_n|| \geq ||x_o + \frac{1}{n} z - g_n|| - \frac{1}{n} \phi_n(z), \qquad (3.3.10)$$

where

$$\phi_n(x_o + \frac{1}{n} z - g_n) = ||x_o + \frac{1}{n} z - g_n||, \text{ and } ||\phi_n|| = 1 \quad (3.3.11)$$

Let $X_o = [x_o - y_o, z]$, the closed linear span of $x_o - y_o$ and z. Since $|\phi_n(z)| \leq ||z||$, it can be assumed that $\{\phi_n(z)\}$ is convergent. Also we see that

$$\lim_{n \to \infty} \phi_n(x_o + \frac{1}{n} z - g_n) = \lim_{n \to \infty} \phi_n(x_o - y_o) = ||x_o - y_o||.$$

Thus we can define a bounded linear functional ϕ_o on X_o by the relation $\phi_o(x) = \lim_{n \to \infty} \phi_n(x)$ for all $x \in X_o$. It is clear that $||\phi_o||_{X_o} = 1$ and $\phi_o(x_o - y_o) = ||x_o - y_o||$. Extend ϕ_o, by Hahn Banach theorem, to the whole of X with preservation of the norm. Clearly then ϕ_o is a subgradient of the norm at $x_o - y_o$. Now, from (3.3.9) and (3.3.10), we obtain

$$F'_K(x_o;z) \leq \phi_n(z) \quad , \text{ for all } n \geq 1$$

and hence $F'_K(x_o;z) \leq \phi_o(z)$ (3.3.12)

But $||x_o - y_o + \frac{1}{n} z|| \geq ||x_o - y_o|| + \frac{1}{n}\phi_o(z)$, for all $n \geq 1$,

$$\text{and hence } F'_K(x_o;z) \leq \lim_{n \to \infty} \frac{||x_o-y_o+\frac{1}{n}z|| - ||x_o-y_o||}{\frac{1}{n}}$$

$$= \lim_{n \to \infty} \frac{f_{y_o}(x_o + \frac{1}{n}z) - f_{y_o}(x_o)}{\frac{1}{n}} = f'_{y_o}(x_o;z) \qquad (3.3.13)$$

This establishes (3.3.7) and the proof is complete.

Example 3.3.11. The formula (3.3.4) may also hold even if K is not M-compact. An example of such a set is the unit ball $U(X)$ of a normed linear space X which has the property (M). In view of Theorem 2.2.2, we need only check the case when $x_o = \theta$. Taking $g_n = -z/||z||$ for all n, we see that $g_n = q(\frac{1}{n}z)$ and

$$F'_K(\theta;z) \leq \frac{||\frac{1}{n}z + \frac{z}{||z||}|| - 1}{\frac{1}{n}} = \frac{f_{-z/||z||}(\frac{1}{n}z) - f_{-z/||z||}(\theta)}{\frac{1}{n}},$$

where $K = U(X)$. Thus, on taking the limit, we get $F'_K(\theta;z) \leq f'_{-z/||z||}(\theta;z)$, q.e.d.

3.4 Local Continuity Behavior of the Farthest Point Map.

In this section we shall consider the local continuity behavior of the farthest point map and some of its consequences.

Theorem 3.4.1. Let X be a reflexive Banach space having the property (M), and let K be any bounded and closed subset of X. Then there exists a subset G dense in X such that

(a) if $x \in G$, then every maximizing sequence in K for x is compact in K, and

(b) the farthest point map q restricted to G is upper semi-continuous.

Proof : By Theorem 3.2.3, there exists a subset D dense in X such that every point $x \in D$ admits atleast one farthest point in K. If $y \in q(x)$, then every point of the half ray $\{\lambda x + (1-\lambda)y : \lambda > 1\}$ admits y as a farthest point in K, because

$$B[\lambda x + (1-\lambda)y, \lambda||x-y||] \supset B[x,||x-y||],$$

and y is a common point to the boundaries of both these balls. Since D is dense in X, so also is the set G, the union of all half rays of the form $\{\lambda x+(1-\lambda)y : \lambda > 1\}$, where $x \in D$ and $y \in q(x)$. Moreover, as a consequence of Theorem 2.2.2, every maximizing sequence $\{g_n\}$ in $B[x,F_K(x)]$ for an element $\lambda x+(1-\lambda)y, \lambda > 1\}$ is compact in $B[x,F_K(x)]$. Since K is a closed subset of $B[x,F_K(x)]$, and y in K is farthest to $\lambda x+(1-\lambda)y$, this is also true for $\{g_n\}$ in the set K. Thus (a) follows. To prove that $q|_G$ is u.s.c., take $x \in G$ and any open set $W \supset q(x)$ and then consider the set

$$M = G \cap \{z \in X : q(z) \subset W\} = \{z \in G : q(z) \subset W\}.$$

Following the argument given in Proposition 3.3.7, it can be easily seen that M is open in G. This proves the upper semi-continuity of q at x and hence (b) follows.

Theorem 3.4.2. Let X be a locally uniformly convex Banach space, and let K be a subset of X having unique farthest point property. Then the farthest point map supported by K is continuous on a dense subset of X.

Proof : Let $q:X \to K$ be the farthest point map supported by K. By assumption it is singlevalued map. Now consider the set $G = \bigcup_{x \in X} \{\lambda x + (1-\lambda)q(x) : \lambda > 1\}$. Clearly G is a dense subset of X. By Corollary 2.2.3, every maximizing sequence $\{g_n\} \subset B[x, F_K(x)]$ for the element $\lambda x + (1-\lambda)q(x), \lambda > 1$ converges to $q(x)$. This is also true if $\{g_n\} \subset K$ and hence if we take $\{x_n\} \subset X$, $x_n \to \lambda x + (1-\lambda)q(x)$, then $\{q(x_n)\}$ is a maximizing sequence in K for $\lambda x + (1-\lambda)\, q(x)$ and $q(x_n) \to q(x)$. But $q(x) \in K$ is farthest to $\lambda x + (1-\lambda)q(x)$, $\lambda > 1$ and hence the result follows:

Theorem 3.4.3. Let X be a reflexive locally uniformly convex Banach space and let K be a bounded, closed set of X. If F_K is Frechet differentiable in a neighbourhood N_x of x, then the restriction of the farthest point map to N_x is singlevalued and continuous.

Proof : By Lemma 3.2.2, $||F_K'(z)|| = 1$ on a subset dense in N_x

and then the continuity of the Frechet derivative of the convex function F_K implies that $||F_K'(z)|| = 1$ on the whole of N_x. Proceeding as in Theorem 3.2.3, we see that every point z in N_x admits a farthest point in K.

To prove the uniqueness of the farthest point we take any y in $q(z)$, where $z \in N_x$ and set

$$b(z,y) = \frac{z-y}{F_K(z)}, \text{ and } D(v) = \{\phi \in S(X^*) : \phi(v) = 1\}, \quad \text{for } v \in S(X). \tag{3.4.1}$$

If $\phi \in D(b(z,y))$, then $\phi(z-y) = F_K(z)$ and hence for $w \in X$,

$$F_K(z) = \phi(z-w+w-y) \leq ||w-y|| + \phi(z-w) \leq F_K(w) + \phi(z-w),$$

$$\text{that is, } F_K(w) \geq F_K(z) + \phi(w-z). \tag{3.4.2}$$

This gives $\phi \in \partial F_K(z) = F_K'(z)$ and hence $D(b(z,y)) = F_K'(z)$ for all $y \in q(z)$. Thus the linear functional $F_K'(z)$ supports the unit sphere at the points $b(z,y), y \in q(z)$. As X is strictly convex, this is possible only when $b(z,y)$ does not depend on y, that is, q is singlevalued in N_x.

Now let $\{x_n\} \subset N_x$ and $x_n \to x_o$ with $x_o \in N_x$. Then $F_x'(x_n) \to F_K'(x_o)$. We shall now denote $b(z) = \dfrac{z-q(z)}{F_K(z)}$ for $z \in N_x$. Then

$$2 = \lim_{n \to \infty} F_K'(x_n)(b(x_n)+b(x_o)) \leq \liminf_{n \to \infty} ||b(x_n) + b(x_o)|| \leq$$

$$\limsup_{n \to \infty} ||b(x_n) + b(x_o)|| \leq 2,$$

and hence $||b(x_n) + b(x_o)|| \to 2$. By the local uniform convexity of the norm $b(x_n) \to b(x_o)$, that is, $q(x_n) \to q(x_o)$. This proves the theorem.

Theorem 3.4.4. Let X be a smooth normed linear space, and let K be a subset of X having unique farthest point property. Then at a point x of continuity of the farthest point map, the function F_K is Gateaux differentiable and the derivative $F'(x;z) = G(x-q(x);z)$, for all $z \in X$.

Proof : For $z \in X$ and $t \in R$, it follows, in view of $G(x;x) = ||x||$ and $G(x;y) \leq ||y||$ for all $x \neq \theta$ and $y \in X$, that

$$||x-q(x+tz)|| \geq ||x+tz-q(x+tz)|| + G(x+tz-q(x+tz);-tz),$$

that is,

$$F_K(x+tz) \leq F_K(x) + G(x+tz-q(x+tz);\ tz). \qquad (3.4.3)$$

By interchanging x and $x+tz$, we get, similarly

$$F_K(x) \leq F_K(x+tz) + G(x-q(x);\ -tz),$$

that is,

$$F_K(x+tz) \geq F_K(x) + G(x-q(x);\ tz) \qquad (3.4.4)$$

As the G-derivative of the norm is norm-to-weak* continuous [7], letting $t \to 0$, we obtain from (3.4.3) and (3.4.4)

$$F_K'(x;z) = G(x-q(x);z),\ z \in X,$$

where $F'_K(x;z) = F'_K(x)(z)$, the value of $F'_K(x)$ at z. Hence the theorem is proved.

If the space X is not necessarily smooth, then F_K is still differentiable in the direction of a certain vector z (see also [8]). This is shown in the following theorem.

Theorem 3.4.5. Let X be a normed linear space, and let K be a subset of X having unique farthest point property. Then at a point x of continuity of the farthest point map

$$\lim_{t \to 0} \frac{F_K(x + t(x-q(x))) - F_K(x)}{t} = F_K(x)$$

Proof: Let $\{t_n\}$ be sequence of positive numbers converging to zero. Then for $z \in X$

$$||x-q(x+t_n z)|| \geq ||x+t_n z - q(x+t_n z)|| - t_n \phi_n(z), \tag{3.4.5}$$

where

$$\phi_n(x+t_n z-q(x+t_n z)) = ||x+t_n z-q(x+t_n z)||, \text{ and } ||\phi_n|| = 1. \tag{3.4.6}$$

The relation (3.4.5) yields

$$F_K(x+t_n z) \leq F_K(x) + t_n \phi_n(z). \tag{3.4.7}$$

By interchanging x and $x + t_n z$, we get, similarly

$$F_K(x+t_n z) \geq F_K(x) + t_n \phi'_n(z), \tag{3.4.8}$$

where

$$\phi_n'(x-q(x)) = ||x-q(x)||, \text{ and } ||\phi_n'|| = 1 \tag{3.4.9}$$

From

$$|\phi_n(x+t_n z - q(x+t_n z)) - \phi_n(x-q(x))| \leq ||t_n z|| + ||q(x+t_n z) - q(x)||,$$

and because of continuity of q, it follows that $\phi_n(x-q(x)) \to ||x-q(x)||$. The relations (3.4.7) and (3.4.8) yield

$$\phi_n'(z) \leq \frac{F_K(x + t_n z) - F_K(x)}{t_n} \leq \phi_n(z)$$

Now taking $z = x-q(x)$, we obtain from the above

$$\lim_{t_n \to +0} \frac{F_K(x + t_n z) - F_K(x)}{t_n} = F_K(x).$$

It can be easily seen that the same result holds if we take t_n negative. Moreover, $\{t_n\}$ being arbitrary, we obtain the required result. This proves the theorem.

Theorem 3.4.6. Let X be a locally uniformly convex Banach space, and let $K \subset X$ have unique farthest point property. If $x_o \in X$ is a point of continuity of the farthest point $q: X \to K$, then every maximizing sequence in K for x_o converges to $q(x_o)$.

Proof: Let $\{g_n\} \subset K$ and $||x_o - g_n|| \to F_K(x_o)$. By Hahn Banach theorem, there exists a $\psi_n \in S(X^*)$ such that

$$\psi_n(x_o - g_n) = ||x_o - g_n||. \tag{3.4.10}$$

Let $t_n^2 = F_K(x_o) - ||x_o - g_n||$ and $t_n \leq 0$. We can assume that

$t_n < 0$, if necessary, by passing onto a subsequence. Then

$$F_K(x_o+t_n(x_o-q(x_o))) \geq ||x_o+t_n(x_o-q(x_o)) - g_n||$$

$$\geq ||x_o-g_n|| + t_n \psi_n(x_o-q(x_o)) \text{ (by (3.4.10))}$$

$$= F_K(x_o) - t_n^2 + t_n \psi_n(x_o-q(x_o)),$$

that is, $$\frac{F_K(x_o+t_n(x_o-q(x_o))) - F_K(x_o)}{t_n} \leq -t_n + \psi_n(x_o-q(x_o))$$

$$\leq -t_n + F_K(x_o). \qquad (3.4.$$

Now applying Theorem 3.4.5, we obtain $\lim_{n\to\infty} \psi_n(x_o-q(x_o))=F_K(x_o)$.

Set

$$z_n = \frac{x_o-g_n}{||x_o-g_n||} \text{ and } z = \frac{x_o-q(x_o)}{||x_o-q(x_o)||}.$$

Then $2 = \lim_{n\to\infty} \psi_n(z_n+z) \leq \liminf_{n\to\infty} ||z_n+z|| \leq \limsup_{n\to\infty} ||z_n+z|| \leq 2$,

and hence $||z_n+ z|| \to 2$. By the local uniform convexity of the norm, $z_n \to z$ and this implies that $g_n \to q(x_o)$. Hence the theorem is proved.

3.5 <u>A Special Case of the Unsolved Problem.</u> We have already mentioned that the Problem 3.1.1 has been solved in the particular case when the farthest point map is continuous. As can be easily seen, this includes the case of M-compact sets having unique farthest point property. We shall now consider a special case

of the Problem 3.1.1 which will also include M-compact sets. In order to bring a larger class of sets than the class of M-compact sets under the purview of this special case, we introduce below a weaker form of upper semi-continuity for the setvalued farthest point map.

<u>Definition</u> 3.5.1. The setvalued farthest point map $q: X \to K$ is said to be Inner Radially Upper Semi-continuous (IRU) at $x_0 \in X$ if, for each $v_0 \in q(x_0)$ and an open set $W \supset q(x_0)$, there exists a neighbourhood N_{x_0} of x_0 such that $W \supset q(x)$ for every $x \in N_{x_0} \cap \{v_0 + \lambda(x_0 - v_0): 0 \leq \lambda \leq 1\}$. The map q is said to be IRU on X if it is IRU at every point of X.

We shall say that a normed linear space "admits centres" if for every bounded, nonempty set K of X, the set $E(K)$, defined in Section 3, is nonempty. It is known that [38] all conjugate Banach spaces, the space $L'(\mu)$ of absolutely integrable functions and the space $C_R(\Omega)$ of realvalued, bounded continuous functions, where Ω is paracompact, admit centres.

<u>Theorem</u> 3.5.2. Let X be a normed linear space admitting centres, and let K be a nonempty subset of X having unique farthest point property. If the farthest point map $q: X \to K$ is IRU continuous on $K + r(K)U(X)$, then K consists of a single point.

<u>Proof</u>: Suppose that K is not a singleton. We may assume that $\theta \in E(K)$. Then as K has unique farthest point property,

of the Problem 3.1.1 which will also include M-compact sets. In order to bring a larger class of sets than the class of M-compact sets under the purview of this special case, we introduce below a weaker form of upper semi-continuity for the setvalued farthest point map.

Definition 3.5.1. The setvalued farthest point map $q:X \to K$ is said to be Inner Radially Upper Semi-continuous (IRU) at $x_0 \in X$ if, for each $v_0 \in q(x_0)$ and an open set $W \supset q(x_0)$, there exists a neighbourhood N_{x_0} of x_0 such that $W \supset q(x)$ for every $x \in N_{x_0} \cap \{v_0 + \lambda(x_0 - v_0): 0 \leq \lambda \leq 1\}$. The map q is said to be IRU on X if it is IRU at every point of X.

We shall say that a normed linear space "admits centres" if for every bounded, nonempty set K of X, the set $E(K)$, defined in Section 3, is nonempty. It is known that [38] all conjugate Banach spaces, the space $L^1(\mu)$ of absolutely integrable functions and the space $C_R(\Omega)$ of realvalued, bounded continuous functions, where Ω is paracompact, admit centres.

Theorem 3.5.2. Let X be a normed linear space admitting centres, and let K be a nonempty subset of X having unique farthest point property. If the farthest point map $q:X \to K$ is IRU continuous on $K+r(K)U(X)$, then K consists of a single point.

Proof: Suppose that K is not a singleton. We may assume that $\theta \in E(K)$. Then as K has unique farthest point property,

there exists an element x of K in the interior of the ball $B[\theta,r(K)]$. So $\theta \varepsilon \operatorname{int} B[x,r(K)] \subset K + r(K)U(X)$. Denote

$$x_o = q(\theta), \text{ and } g_n = q(\tfrac{1}{n} x_o). \tag{3.5.1}$$

Then taking a ϕ_n in $S(X^*)$ such that

$$\phi_n(g_n - \tfrac{1}{n} x_o) = ||g_n - \tfrac{1}{n} x_o|| , \tag{3.5.2}$$

we obtain

$$||g_n|| \geq \phi_n(g_n - \tfrac{1}{n} x_o + \tfrac{1}{n} x_o) = ||g_n - \tfrac{1}{n} x_o|| + \tfrac{1}{n} \phi_n(x_o). \tag{3.5.3}$$

Since $B[\theta,r(K)]$ is the minimal ball containing K, we have by (3.5.3)

$$0 < ||g_n - \tfrac{1}{n} x_o|| - ||g_n|| \leq -\tfrac{1}{n} \phi_n(x_o),$$

that is $\phi_n(x_o) < 0$. (3.5.4)

By the IRU continuity of the map q, $g_n \to x_o$. Hence

$\lim_{n \to \infty} \phi_n(g_n - \frac{1}{n} x_o) = \lim_{n \to \infty} \phi_n(x_o) = ||x_o||$. But, by (3.5.4), $\lim_{n \to \infty} \phi_n(x_o) \leq 0$ and this gives a contradiction since $|| x_o|| = 1$.

Remark 3.5.3. The proof given above differs from that of Baltter's ([8] , Theorem 2 and 4) in two respects. First, X need not be complete (provided there exists an incomplete

normed linear space admitting centres) and secondly, the proof does not depend on the axiom of choice.

Corollary 3.5.4. Let K be a nonempty M-compact set of a normed linear space X admitting centres. If K has unique farthest point property, then it consists of a single point.

Proof: The result follows from Theorem 3.5.2 since K supports a continuous farthest point map.

CHAPTER - IV

APPROXIMATIVE COMPACTNESS AND CONTINUITY OF METRIC PROJECTIONS

4.1 Introduction. The concept of approximative compactness of a set has been introduced by Efimov and Stechkin in [33]. It is known that every boundedly compact set in a normed linear space and every weakly sequentially closed set in a uniformly convex Banach space are approximatively compact [33]. Singer [71] has characterized those spaces which share this property of uniformly convex Banach spaces. In the same paper, it is proved that an approximatively compact Chebyshev set supports a continuous metric projection. In the present chapter, we show that though, in general, the continuity of the metric projection does not imply the approximative compactness of the supporting set, it does so in the setting of a space with the property (M). It is also shown that in a space X with the property (M), the metric projection supported by a Chebyshev set is continuous on a subset dense in X.

4.2 Approximative Compactness and Continuity of Metric Projections.

We now prove the equivalence of the notions of approximative compactness of a set and the continuity of the associated metric projection in a space with the property (M).

Theorem 4.2.1. Let K be a Chebyshev subset of a normed linear space X having the property (M). Then K is approximatively

compact if and only if the metric projection P_K is continuous.

For the proof of the theorem we need the following lemma due to Vlasov ([79] , Lemma 1), a proof of which is given below.

Lemma 4.2.2. Let X be a normed linear space, and let K be a Chebyshev subset of X. If $x \in X$ is a point of continuity of the metric projection P_K, then

$$\lim_{t \to 0} \frac{d_K(x + t(x - P_K(x))) - d_K(x)}{t} = d_K(x),$$

where $d_K(X) = \inf \{||x-y|| : y \in K\}$.

Let $z \in X$ and $t \in R$. There exists a $\phi_t \in S(X^*)$ such that

$$\phi_t(x-P_K(x + tz)) = ||x - P_K(x + tz)|| \tag{4.2.1}$$

Then $||x + tz - P_K(x + tz)|| \geq ||x - P_K(x + tz)|| + \phi_t(tz)$

$$\geq d_K(x) + \phi_t(tz),$$

so that $d_K(x + tz) - d_K(x) \geq \phi_t(tz)$. $\qquad (4.2.2)$

Similarly, a $\phi'_t \in S(X^*)$ can be chosen so that

$$\phi'_t(x + tz - P_K(x)) = ||x + tz - P_K(x)||, \tag{4.2.3}$$

and this implies that

$$||x - P_K(x)|| \geq ||x + tz - P_K(x)|| - \phi'_t(tz)$$

$$\geq d_K(x + tz) - \phi'_t(tz),$$

and hence $d_K(x + tz) - d_K(x) \leq \phi'_t(tz)$. $\qquad (4.2.4)$

From

$$|\phi_t(x-P_K(x+tz)) - \phi_t(x-P_K(x))| \leq ||P_K(x+tz) - P_K(x)||,$$

$$|\phi_t'(x+tz-P_K(x)) - \phi_t'(x-P_K(x))| \leq |t| \cdot ||z|| ,$$

and the continuity of P_K at x, it follows that

$$\lim_{t \to 0} \phi_t(x - P_K(x)) = ||x - P_K(x)|| = \lim_{t \to 0} \phi_t'(x-P_K(x)).$$

Thus taking $z = x-P_K(x)$, we obtain from (4.2.2) and (4.2.4) the required result.

<u>Proof of Theorem</u> 4.2.1. Let $\{g_n\} \subset K$, $x \in X \sim K$ and $||x - g_n|| \to d_K(x)$. Suppose that P_K is continuous. Setting $t_n = (||x-g_n|| - d_K(x))^{1/2}$, we see that $t_n \to 0$ through non-negative values. We can assume without any loss of generality that $t_n > 0$ by passing to a subsequence, if necessary. Choose $\phi_n \in S(X^*)$ such that

$$\phi_n(x+t_n(x-P_K(x))-g_n) = ||x+t_n(x-P_K(x))-g_n||, \tag{4.2.5}$$

and then it follows that

$$\begin{aligned} ||x-g_n|| &\geq ||x+t_n(x-P_K(x)) - g_n|| - t_n \phi_n(x-P_K(x)) \\ &\geq d_K(x+t_n(x-P_K(x))) - t_n \phi_n (x - P_K(x)). \end{aligned}$$

Thus $t_n^2 + d_K(x) \geq d_K(x + t_n(x-P_K(x))) - t_n \phi_n(x - P_K(x))$,

that is,
$$\begin{aligned} \frac{d_K(x+t_n(x-P_K(x))) - d_K(x)}{t_n} &\leq t_n + \phi_n(x - P_K(x)) \\ &\leq t_n + ||x-P_K(x)|| \end{aligned} \tag{4.2.6}$$

Applying Lemma 4.2.2 to (4.2.6), we see that $\lim_{n\to\infty} \phi_n(x - P_K(x)) = ||x - P_K(x)||$. Set

$$z_n = \frac{x+t_n(x-P_K(x)) - g_n}{||x+t_n(x-P_K(x)) - g_n||} \text{ and } z = \frac{x-P_K(x)}{||x-P_K(x)||}.$$

Then $2 = \lim_{n\to\infty} \phi_n(z_n+z) \le \liminf_{n\to\infty} ||z_n+z|| \le \limsup_{n\to\infty} ||z_n+z|| \le 2$,

and hence $||z_n + z|| \to 2$. Since X has the property (M), the sequence $\{z_n\}$ has a convergent subsequence. Moreover, it follows from

$$\big|\, ||x + t_n(x-P_K(x)) - g_n|| - ||x-g_n||\, \big| \le t_n\, ||x-P_K(x)||$$

that $||x + t_n(x - P_K(x))-g_n|| \to d_K(x)$, where $x \in X \sim K$. This shows that $\{g_n\}$ has a convergent subsequence.

The converse is well-known and holds even in a metric space setting (see [71]).

The following example shows that, in general, continuity of the metric projection does not imply that the Chebyshev set is approximatively compact.

Example 4.2.3. Let X^* be the dual space of the Banach space constructed by Klee ([53], see also Proposition 2.1.4) by suitably renorming ℓ^2. The norm in X has the property that it is Gateaux smooth and is Frechet smooth except at two points. Thus X^* is a reflexive, strictly convex Banach space lacking

the property (h), since otherwise X would be strongly smooth. In other words, X^* does not satisfy the Efimov Stechkin property. By Theorem 2.1.1, X^* contains a closed hyperplane K which is not approximatively compact. However, in view of the reflexivity and the strict convexity of X^*, the hyperplane K is a Chebyshev set. By a result of Lambert (see Holmes [38] , page 165), every Chebyshev subspace in X^* supports a continuous metric projection. Thus the hyperplane K supports a continuous metric projection, but is not approximatively compact. The space X^*, of course, lacks the property (M) in as much as it lacks the property (h) (see Theorem 2.3.1).

The above example shows that there is a need to characterize spaces in which every proximal convex set and, more generally, every proximal sun is approximatively compact. It may be noted that sun is a generalization of a convex set. Further, various generalizations of a sun have also been known (see Chapter 1). It has been already mentioned that (see Remark 1.6.1) in a Banach space X with the property (M), the concepts of α_1, α, β, γ, δ - suns are the same. It is our objective to consider such type of suns and the following theorem will show that spaces with the property (M) provide a nice setting in dealing with such type of problems.

<u>Theorem</u> 4.2.4. In a normed linear space with the property (M), every proximal α_1-sun is approximatively compact.

Proof : Let K be a proximal α_1-sun (see 1.6, Chapter 1), and let $x \in X \sim K$. Choose a sequence $\{\varepsilon_n\}$ of real numbers such that $\varepsilon_n > 0$ and $\varepsilon_n \to 0$. From the definition of an α_1-sun, there exists a halfray ℓ_x with vertex x and depending on ε_n, such that

$$d_K(z) \geq ||z-x|| + d_K(x) - \varepsilon_n, \text{ for all } z \in \ell_x.$$

Let z_n be the element in ℓ_x such that $||z_n-x|| = d_K(x)$. Then we obtain

$$d_K(z_n) \geq 2d_K(x) - \varepsilon_n, \text{ for each } n.$$

Let x' be any element of $P_K(x)$; then $d_K(z_n) \leq ||z_n-x'|| \leq 2d_K(x)$. Thus we have

$$\lim_{n \to \infty} d_K(z_n) = \lim_{n \to \infty} ||z_n-x'|| = 2d_K(x). \tag{4.2.7}$$

This shows that $||\frac{z_n - x}{d_K(x)} + \frac{x - x'}{d_K(x)}|| \to 2$, where $||z_n-x|| = d_K(x) = ||x - x'||$. As X has the property (M), the sequence $\{\frac{z_n-x}{d_K(x)}\}$ is compact; that is, the sequence $\{z_n\}$ has a convergent subsequence. Without any loss of generality, we assume that $z_n \to z_o$. Clearly $||z_o - x|| = d_K(x)$.

Let $\{g_n\} \subset K$ and $||x - g_n|| \to d_K(x)$. Suppose that $t_n = (||x - g_n|| - d_K(x))$; then clearly $t_n \to 0$ as $n \to \infty$. If ϕ_n is a subgradient of the norm at $z_n - g_n$, then

$$||x-g_n|| \geq ||z_n-g_n|| + \phi_n(x-z_n),\ ||\phi_n|| = 1 \text{ and } \phi_n(z_n-g_n) = ||z_n-g_n||. \tag{4.2.8}$$

From this we obtain

$$t_n + d_K(x) \geq d_K(z_n) + \phi_n(x-z_n),$$

that is, $d_K(z_n) \leq t_n+d_K(x) + \phi_n(z_n-x) \leq t_n+d_K(x) + ||z_n-x||$

$$= t_n + 2\, d_K(x),$$

and hence $\phi_n(z_n-x) \to d_K(x)$. As $z_n \to z_o$, this shows that $\phi_n(z_o - x) \to d_K(x)$.

Now,

$$2 = \lim_{n\to\infty} \phi_n\left(\frac{z_n-g_n}{||z_n-g_n||} + \frac{z_o-x}{||z_o-x||}\right) \leq \liminf_{n\to\infty} \left|\left|\frac{z_n-g_n}{||z_n-g_n||} + \frac{z_o-x}{||z_o-x||}\right|\right|$$

$$\leq \limsup_{n\to\infty} \left|\left|\frac{z_n-g_n}{||z_n-g_n||} + \frac{z_o-x}{||z_o-x||}\right|\right| \leq 2,$$

whence $\left|\left|\frac{z_n- g_n}{||z_n- g_n||} + \frac{z_o- x}{||z_o-x||}\right|\right| \to 2$ as $n \to \infty$. As X has the property (M), the sequence $\left\{\frac{z_n-g_n}{||z_n-g_n||}\right\}$ has a convergent subsequence. Moreover, $||z_n-g_n|| \geq d_K(z_n)$, $d_K(z_n) \to 2d_K(x)$ and $||z_n- g_n|| \leq ||z_n- x|| + ||x-g_n|| = d_K(x) + ||x-g_n||$, which is bounded. As $z_n \to z_o$, this implies that the sequence $\{g_n\}$ has a convergent subsequence in K. This proves the theorem.

Remark 4.2.5. It is not known if an approximatively compact set, in general, is a sun of some type. The result is known only when it is a Chebyshev set.

Corollary 4.2.6. In a space with the property (M), no Chebyshev sun can support a discontinuous metric projection.

Proof : This follows immediately from the above theorem observing that an α_1-sun is a generalization of a sun.

In the following we use Theorem 2.2.4 to obtain a result about the continuity behavior of the metric projection onto a Chebyshev set which may not be a sun of any type.

Theorem 4.2.7. Let K be a Chebyshev set in a space with the property (M). Then there exists a subset G dense in X such that

(i) if $x \in G$, $\{g_n\} \subset K$ and $||x - g_n|| \to d_K(x)$, then $g_n \to P_K(x)$, and

(ii) the metric projection restricted to G is continuous.

Proof: Since $P_K(x) = x$ for $x \in K$, we need only prove the result in the complement of K. Let $x \in X \sim K$, and $z = \lambda x + (1-\lambda)P_K(x)$, $0 < \lambda < 1$. To prove (i) we consider a sequence $\{g_n\} \subset K$ such that $||z-g_n|| \to d_k(z) = \lambda d_k(x)$. Now set

$$x' = \frac{z-x}{d_K(x)} \quad , \quad g_n' = \frac{g_n-x}{d_K(x)} . \tag{4.2.9}$$

Then, clearly, $0 < ||x'|| < 1$, $||g_n'|| \geq 1$ and $||x' - g_n'|| = ||\frac{z-g_n}{d_K(x)}|| \to \lambda = 1 - ||x'||$. Hence applying Theorem 2.2.4, we see that the sequence $\{g_n'\}$ is compact in X. This proves that the sequence $\{g_n\}$ has a convergent subsequence in K. But K is Chebyshev and hence $P_K(x)$ is the only strong cluster point of the sequence $\{g_n\}$ showing that $g_n \to P_K(x)$. The set $G = K \bigcup_{x \in X \sim K} \{\lambda x + (1-\lambda) P_K(x) : 0 < \lambda < 1\}$ is clearly dense in X and (i) follows.

To prove (ii), take $\{z_n\} \subset G$, $z = \lambda x + (1-\lambda) P_K(x) \in G$, such that $z_n \to z$. Then it follows from

$$\big|\, ||z-P_K(z)|| - ||z_n-P_K(z_n)||\, \big| = |d_K(z) - d_K(z_n)| \leq ||z_n - z|| ,$$

that $\lim_{n \to \infty} ||z_n-P_K(z_n)|| = \lim_{n \to \infty} ||z-P_K(z_n)|| = d_K(z)$

and hence taking $g_n = P_K(z_n)$, we obtain, as in (i), $g_n \to P_K(z)$. This proves the result.

4.3 <u>A Special Case</u>. The assumption in Theorem 4.2.7 that K is Chebyshev can be relaxed if the norm in X has some additional properties.

<u>Theorem</u> 4.3.1. Let K be either (a) a closed subset of a uniformly convex Banach space, or (b) a proximal subset of a locally uniformly convex Banach space. Then there exists a subset G dense in X such that

(i) if $x \in G$, $\{g_n\} \subset K$ and $||x - g_n|| \to d_K(x)$, then the sequence $\{g_n\}$ is convergent in K, and

(ii) the restriction of P_K to G is singlevalued and continuous.

<u>Proof</u>: First, let K be a closed subset of a uniformly convex Banach space. Then by a result of Edelstein [26], there exists a subset D dense in X such that every point in D admits a nearest point in K (see also Stechkin [74]). The union G of all sets

$$\{\lambda x + (1-\lambda) y : 0 \leq \lambda < 1, y \in P_K(x)\}, x \in D,$$

is clearly dense in X. Moreover, if $x \in D$ and $y \in P_K(x)$, then by the strict convexity of the norm

$$S[x, d_K(x)] \cap S[\lambda x + (1-\lambda)y, \lambda d_K(x)] = \{y\}, 0 \leq \lambda < 1,$$

where $S[x,r]$ denotes the sphere centred at x and of radius r. Hence every element of G admits a unique nearest point in K. Now let z be any element of G; then z can be written in the form $z = \lambda x + (1-\lambda)y$, where $x \in D$, $y \in P_K(x)$ and $0 \leq \lambda < 1$. If $\{g_n\} \subset K$ and $||z - g_n|| \to d_K(z) = \lambda d_K(x)$, then setting

$$x' = \frac{z-x}{d_K(x)} \quad \text{and} \quad g_n' = \frac{g_n - x}{d_K(x)},$$

we see that $0 < ||x'|| = 1 - \lambda \leq 1$, $||g_n'|| \geq 1$ and $||x' - g_n'|| = ||\frac{z-g_n}{d_K(x)}|| \to \lambda = 1 - ||x'||$. We can assume that $\lambda \neq 0$, that is,

$x \notin K$, since otherwise (i) would follow trivially. Now, we see that all the conditions of Theorem 2.2.4 are satisfied and hence the sequence $\{g_n'\}$ is compact in X. This shows that the sequence $\{g_n\}$ has a convergent subsequence in K. As z has a unique nearest point in K, this implies that $g_n \to y = P_K(z)$ and thus (i) follows. The proof of (ii) is the same as in Theorem 4.2.7.

In the case of (b), the set D is the whole space X and the proof is the same. Hence the theorem is proved.

CHAPTER - V

ON EQUIDISTANT SETS IN NORMED LINEAR SPACES

5.1 Introduction. For any two distinct points x and y of a normed linear space X, let $E(x,y)$ denote the equidistant set from x and y, that is, the set of points p in X for which $||p-x|| = ||p-y||$. Such sets have been introduced by Kalish and Straus in [46] in connection with the study of "determining" sets in Banach spaces. Recently, it has been shown by Klee [52] that there is a connection between weak continuity of metric projections and weak closure of equidistant sets. Similar results have been proved by Kottman and Lin [54] with respect to weak and bounded weak topologies. There arises now the following problem: When is an equidistant set weakly closed? It is easy to see that every equidistant set in an inner-product space or in a finite dimensional Banach space is weakly closed. But apart from these spaces, does there exist a space in which every equidistant set is weakly or weakly sequentially closed? The object of this chapter is to answer this question. It is proved that in an ℓ^p-space $(1 < p < \infty)$, every equidistant set is bounded weakly closed. On the other hand, no equidistant set in $L^p(\mu)$ $(1 < p < \infty,\ p \neq 2$, and μ a separable nonatomic measure) and c_o is weakly sequentially closed.

5.2 Notation and some basic results. Let M be a Chebyshev subspace of a normed linear space, and let P_M be the metric projection supported by M. The following theorem will be useful:

Theorem 5.2.1. ([38] , page 159). The metric projection P_M supported by a Chebyshev subspace M in a normed linear space X satisfies the following properties:

1) P_M is idempotent and closed;

2) $||P_M(x)|| \leq ||x-P_M(x)|| + ||x|| \leq 2||x||$, for all $x \in X$;

3) P_M is homogeneous (i.e., $P_M(tx) = tP_M(x)$, for all $x \in X$ and $t \in R$);

4) P_M is additive modulo M (i.e., $P_M(x+y) = P_M(x) + P_M(y)$, if either x or $y \in M$).

The kernel of the metric projection P_M is the set $\{x \in X: P_M(x) = \theta\}$ and is denoted by M^θ.

Proposition 5.2.2. ([38], page 160). Let M be a Chebyshev subspace of a normed linear space X. Then $X = M \oplus M^\theta$.

Proposition 5.2.3 ([39] , Proposition 4). Let X be a smooth Banach space. Let $\{M_\alpha\}$ be an arbitrary family of Chebyshev subspaces of X, and suppose that M, the closed linear span of $\{M_\alpha\}$, is also a Chebyshev subspace of X. Then $M^\theta = \bigcap_\alpha M_\alpha^\theta$.

Proposition 5.2.4 ([54]) Let X be a Banach space, and let M be a reflexive, Chebyshev subspace such that M^θ is bw-closed. Then P_M is bw-continuous.

For $x \neq \theta$ in X, let $E(-x,x)$ denote the equidistant set from x and $-x$; i.e., the set of points $y \in X$ such that $||y-x|| = ||y+x||$. Observe that each equidistant set is closed in the norm topology. It can be also easily checked that $E(-x,x) + y = E(-x+y,x+y)$ and $E(-\lambda x,\lambda x) = \lambda E(-x,x)$ for $\lambda \in R$. If x and $y \in X$ and $||x-y|| = ||x+y||$, we say that x is orthogonal to y and write $x \perp y$. Thus $E(-x,x)$ is then the set of all elements in X which are orthogonal to x. This concept of orthogonality is named the isosceles orthogonality and has been studied by James in [40]. We shall need the following result of James [40]:

Proposition 5.2.5. For each pair of linearly independent vectors x and y in X, there exists a number $t \in R$ such that $x + ty \perp x$.

By a cone in X, we shall mean a set K such that $x \in K \Rightarrow tx \in K$, for every $t \geq 0$. Observe that M^{θ} is a cone. The closed linear span of vectors $x_1,x_2,\ldots x_n$ will be denoted by $[x_1,x_2\ldots,x_n]$.

5.3 Some Properties of Equidistant Sets. We shall now study some geometrical and topological properties of equidistant sets.

Lemma 5.3.1. Let x be any nonzero element of a two dimensional normed linear space X. If $E(-x,x)$ is convex, then it is a line through the origin.

Proof : Let $E(-x,x)$ be convex and $z \neq \theta$ be any element in $E(-x,x)$. By Proposition 5.2.5 such a point z exists. We shall show that $E(-x,x) = [z]$ and that will prove the lemma. First, since $E(-x,x)$ is symmetric about the origin, $-z$ is in $E(-x,x)$ and then because of the convexity of the set $E(-x,x)$, $\{tz : |t| \leq 1\} \subset E(-x,x)$. If $y \in E(-x,x)$ is linearly independent from z, then either y and z or $-y$ and z are separated by the line $[x]$. Since $y \in E(-x,x)$ implies $-y \in E(-x,x)$, we assume that the former holds. Then the line segment joining y and z is contained in $E(-x,x)$, but since this line intersects $[x]$ at a point other than the origin it cannot be a point of $E(-x,x)$. Hence there is a contradiction. This means that any two nonzero points of $E(-x,x)$ are linearly dependent.

Now we show that $E(-x,x)$ is unbounded. Let $z \in E(-x,x)$ and $\lambda > 1$ be arbitrary. Then again by Proposition 5.2.5 there exists a $t \in R$ such that $\lambda z + tx \in E(-x,x)$. From what we have already seen, this will mean that z and $\lambda z + tx$ are linearly dependent. This is possible only if $t = 0$. Hence the result is proved.

Theorem 5.3.2. Let $x \neq \theta$ be any element of a normed linear space X. If $E(-x,x)$ is convex, then it is a proximal subspace of co-dimension one.

Proof : Let $E(-x,x)$ be convex and z be any element of X

outside $[x]$. Then $E(-x,x) \cap [x,z]$ is convex and by Lemma 5.3.1 it is a line. Thus if z is an element of $E(-x,x)$ then the span $[z]$ is contained in $E(-x,x)$. Consequently, $E(-x,x)$ is a convex cone symmetric about the origin. If $z_1, z_2 \in E(-x,x)$, then $\lambda z_1 \in E(-x,x)$ and $\mu z_2 \in E(-x,x)$ for all $\lambda, \mu \in R$. Take any $\alpha, \beta \in R$. Since $E(-x,x)$ is assumed to be convex, any convex combination of λz_1 and μz_2 is in $E(-x,x)$ and, in particular, $\frac{1}{2}(\lambda z_1 + \mu z_2) \in E(-x,x)$. We can choose λ, μ such that $\frac{1}{2}\lambda = \alpha$ and $\frac{1}{2}\mu = \beta$ and hence $\alpha z_1 + \beta z_2 \in E(-x,x)$. This proves that $E(-x,x)$ is a subspace. Now let $u \in X$. Then either $u = \lambda x$ or, by Proposition 5.2.5, $u + \lambda x = z \in E(-x,x)$ for some $\lambda \in R$. Thus $E(-x,x)$ and x together span X. Therefore, $E(-x,x)$ is of co-dimension one. Since each equidistant set is closed it follows that $E(-x,x)$ is a closed subspace.

Now let $h \in E(-x,x)$. Then, by the convexity of $E(-x,x)$, $||x-\lambda h|| = ||x+\lambda h||$, $0 \leq \lambda < 1$. As $||x|| \leq ||x-\lambda h|| = ||x+\lambda h||$, we see that θ is a nearest point of x in $E(-x,x)$. If $\alpha \in R$, then

$$||\alpha x-h|| = |\alpha| \ ||x-\alpha^{-1}h|| = |\alpha| \ ||x+\alpha^{-1}h|| = ||\alpha x + h||$$

for all $h \in E(-x,x)$ and hence θ is also a nearest point in $E(-x,x)$ to αx. As any $w \in X$ has a representation $w = \alpha x+h$, where $h \in E(-x,x)$, we have

$$||w-h|| = ||\alpha x|| \leq ||\alpha x-z||, \ z \in E(-x,x)$$

which implies

$$||w-h|| \leq ||w-z-h||, \; z \in E(-x,x).$$

But $E(-x,x)$ being a subspace, $z + h \in E(-x,x)$ and hence $||w-h|| \leq ||w-v||$ for every $v \in E(-x,x)$. Thus every $w \in X$ has a nearest point in $E(-x,x)$, that is, $E(-x,x)$ is proximal.

Day [20, Theorem 5.4] has shown that if every equidistant set in a normed linear space is a linear manifold then the space must be an inner-product one. Now in view of Theorem 5.3.2 this result can be stated in the following form:

Theorem 5.3.3. Let X be a normed linear space. If $E(-x,x)$ is convex for each $x \in X$, then X is an inner-product space.

Proof : Since each $E(-x,x)$ is a linear manifold, by Theorem 5.3.2, the result follows.

Theorem 5.3.4. Let M be the one dimensional span $[x]$ of x in a normed linear space X. Let M be Chebyshev. Then the following hold:

(1°) $M^\theta \subset E(-x,x) \Longrightarrow M^\theta = E(-x,x)$.

(2°) $E(-x,x)$ is a cone $\Longrightarrow M^\theta = E(-x,x)$.

The following lemma will be needed in the proof of the theorem:

Lemma 5.3.5 ([54], Lemma 1). If $x \in X$ and $M = [x]$ is Chebyshev, then

$$P_M(E(-x,x)) \subset \{tx : -1 \leq t \leq 1\}.$$

Proof of the theorem.

(1°) If $u \in E(-x,x)$, then by the above lemma $P_M(u) = \alpha x$ with $|\alpha| \leq 1$. Since $||u-x|| = ||u+x||$ and u has a unique nearest point in $[x]$, $|\alpha| \neq 1$. We can write, in view of Proposition 5.2.2, $u = u_\theta + \alpha x$, where $u_\theta \in M^\theta$ and, since $\lambda u_\theta \in M^\theta \subset E(-x,x)$ for all $\lambda \in R$ (M^θ being a cone), we have

$$||\lambda u_\theta - x|| = ||\lambda u_\theta + x|| ,$$

which is the same as

$$||u_\theta - \mu x|| = ||u_\theta + \mu x|| , \text{ for } \mu \in R. \tag{5.3.1}$$

In particular, with $\mu = 1-\alpha$, we have

$$||u + (1-2\alpha)x|| = ||u_\theta + \alpha x + (1-2\alpha)x|| = ||u_\theta + \alpha x + (\mu-\alpha)x||$$

$$= ||u_\theta + \mu x|| = ||u_\theta - \mu x|| = ||u-x|| = ||u+x|| \tag{5.3.2}$$

$= r$, say.

So the sphere centred at u and of radius r meets M in atleast three points : $-x$, x, and $(-1+2\alpha)x$, which is impossible

unless $1-2\alpha = \pm 1$. Thus $\alpha = 0$ or $\alpha = 1$ and the latter, we saw above, is also impossible. Therefore $\alpha = 0$ and $u \in M^{\theta}$.

(2°) Let $u \in M^{\theta}$. Then there exists a number t such that $u-tx \in E(-x,x)$. By Theorem 5.2.1, $P_M(u-tx) = P_M(u) + P_M(-tx) = -tx$ and since $u-tx \in E(-x,x)$, applying Lemma 5.3.5, we get $|t| \leq 1$. By assumption $E(-x,x)$ is a cone and hence $\lambda u - \lambda tx \in E(-x,x)$. As $\lambda u \in M^{\theta}$, for all $\lambda \in R$, this again means that $|\lambda t| \leq 1$ for all $\lambda \in R$. This is possible only when $t = 0$. Hence $M^{\theta} \subsetneq E(-x,x)$ and by (1°) above $M^{\theta} = E(-x,x)$.

We see from (2°) that as cones the kernel M^{θ} and the set $E(-x,x)$ coincide. However, M^{θ} may even be a subspace but $E(-x,x)$ is not. In fact, Kottman and Lin [54] have given an example where M^{θ} is a closed hyperplane, but $E(-x,x)$ is not even weakly sequentially closed. In the following we examine their relationship as regards weak topology (see also [54]).

Theorem 5.3.6. Let $M = [x]$ be a Chebyshev subspace of a normed linear space X. Then M^{θ} is weakly (bounded weakly, or weakly sequentially) closed if $E(-x,x)$ is weakly (bounded weakly, or weakly sequentially) closed.

Proof : We consider the case when $E(-x,x)$ is weakly closed, the other cases being similar. Let $\{u_{\alpha}\} \subset M^{\theta}$ be a net which

converges weakly to an element $u \in X$. Suppose that $u \notin M^{\theta}$; that is, $P_M(u) \neq \theta$. If we denote $P_M(u) = 2z$, then by Proposition 5.2.5 there exists a net $\{t_\alpha\}$ of real numbers such that $u_\alpha - t_\alpha z \in E(-z,z)$ and $|t_\alpha| \leq 1$. A subnet $\{t_\beta\}$ can be extracted such that it converges to a number t_o with $|t_o| \leq 1$. As z is a scalar multiple of x, the set $E(-z,z)$ is also a scalar multiple of $E(-x,x)$ and hence is weakly closed. Consequently, $u - t_o z \in E(-z, z)$ and $P_M(u - t_o z) = 2z - t_o z \in \{tz : |t| \leq 1\}$. This shows that $1 \leq t_o \leq 3$ and since $|t_o| \leq 1$, we obtain $t_o = 1$. Thus $u-z \in E(-z,z)$ and hence $||u-2z|| = ||u- P_M(u)|| = ||u-\theta||$. This contradicts the Chebyshev property of M and the theorem is proved.

We shall now consider a structural property of the set $E(-x,x)$.

<u>Theorem</u> 5.3.7. Let $E(-x,x)$ be a convex subset of a normed linear space X with $||x|| = 1$. Then $E(-x,x)$ is Chebyshev if and only if x is an extreme point of the unit ball of X.

<u>Proof</u> : Let $E(-x,x)$ be a Chebyshev set. It will be actually a subspace because of Theorem 5.3.2. If x is not an extreme point of the unit ball of X, then there exists a pair of points x_1 and x_2 in the unit sphere $S(X)$ such that $x = \frac{1}{2}(x_1 + x_2)$ and $I = \{tx_1 + (1-t)x_2 : 0 \leq t \leq 1\}$ is contained in $S(X)$. Now

$$||x_1 - x - x|| = ||x_2|| = 1 = ||x_1|| = ||x_1 - x + x||$$

and hence $x_1 - x \in E(-x,x)$. Similarly $x_2 - x \in E(-x,x)$. Thus $x_1, x_2 \in E(\theta, 2x)$ and since $E(-x,x)$ is a subspace, $I \subset E(\theta,2x)$. As in Theorem 5.3.2, θ is a nearest point in $E(-x,x)$ to x. It is also unique because by assumption, $E(-x,x)$ is Chebyshev. This in turn implies that θ has the nearest point x in $E(\theta,2x)$. But $I \subset E(\theta,2x)$ and every element of I is at unit distance from θ. This contradicts the Chebyshev property of $E(\theta,2x)$.

Conversely, let x be an extreme point of the unit ball $U(X)$. We have already seen that θ in $E(-x,x)$ is nearest to x. If there exists an $y \in E(-x,x)$ such that $y \neq \theta$ but is nearest to x, then every point λy with $0 \leq \lambda \leq 1$ is also nearest to x, that is, $||x-\lambda y|| = 1$ for $0 \leq \lambda \leq 1$. But $-y$ also belongs to $E(-x,x)$ and hence $||x-\lambda y|| = 1$ for $-1 \leq \lambda \leq 1$. This contradicts the fact that x is an extreme point. Hence θ in $E(-x,x)$ is the unique nearest point of x. If $h \in E(-x,x)$ is nearest to αx, $\alpha \in R$, then $||\alpha x - h|| \leq ||\alpha x - \theta||$, i.e., $||x - \frac{h}{\alpha}|| \leq ||x||$. But $h/\alpha \in E(-x,x)$ and the above argument shows that $h = \theta$. Hence θ is nearest to every αx, $\alpha \in R$. By theorem 5.3.2, $X = E(-x,x) \oplus [x]$ and hence any $u \in X$ can be written in the form $u = z + \lambda x$, where $z \in E(-x,x)$ and $\lambda \in R$. Now

$$\inf_{v \in E(-x,x)} ||u-v|| = \inf_{v \in E(-x,x)} ||\lambda x+z-v|| = \inf_{y \in E(-x,x)} ||\lambda x-y||$$

$$= ||\lambda x|| = ||u-z||. \qquad (5.3.3)$$

This shows that z is a nearest point of u in $E(-x,x)$. That it is the unique nearest point of u follows from the fact that if $||u-z_1|| = ||u-z||$, $z_1 \in E(-x,x)$, then

$$||\lambda x|| = ||z-z_1+ \lambda x|| = ||\lambda x - (z_1-z)||,$$

and $z_1-z \in E(-x,x)$. As θ is the unique nearest point of λx, $z = z_1$ and this proves the result.

We illustrate 5.3.7 by the following examples:

Example 5.3.8. Let $X = R^2$ be equipped with the supremum norm, and let $x=(1,1)$ and $z = (-1,1)$. Then

$$E(-x,x) = \{(z_1,z_2) \in R^2:\max \{|z_1-1|,|z_2-1|\} = \max \{|z_1+1|,|z_2+1|\}\}.$$

Considering various possibilities we obtain the following:

(i) $|z_1-1| = |z_1+1| \Longrightarrow z_1 = 0$; $|z_2 \pm 1| \leq 1 \Longrightarrow z_2 = 0$

(ii) $|z_1-1| = |z_2+1| \Longrightarrow z_1 + z_2 = 0$

(iii) $|z_2-1| = |z_1+1| \Longrightarrow z_1 + z_2 = 0$

(iv) $|z_2-1| = |z_2+1| \Longrightarrow z_2 = 0$; $|z_1 \pm 1| \leq 1 \Longrightarrow z_1 = 0$.

Thus $E(-x,x) = \{(z_1,z_2) \in R^2: \ z_1 + z_2 = 0\} = [z]$. Similarly, it can be shown that $E(-z,z) = [x]$. To prove that $[x]$ is Chebyshev, it is sufficient, in view of $X = [x] \oplus [z]$, to show that z has a unique nearest point in $[x]$. Indeed,

$$\inf_{\lambda \in R} ||z-\lambda x|| = \inf_{\lambda \in R} \max \{|1-\lambda|,|1+\lambda|\} = \inf_{\lambda \geq 0} (1+\lambda) = 1 = ||z||,$$

and the infimum attains only at $\lambda = 0$. Thus θ is the unique nearest point in $[x]$ to z. Hence $[x]$ is Chebyshev. Similarly, $[z]$ is also Chebyshev. Clearly, x and z both are extreme points of the unit ball. It is also clear that if $M_x = [x]$ and $M_z = [z]$, then $M_x^\theta = [z]$ and $M_z^\theta = [x]$.

<u>Example</u> 5.3.9. Let $X = \ell^1$ and e_i be the vector such that $e_i(j) = \delta_{ij}$, the Kronecker delta. Then $E(-e_i,e_i) = \{z \in \ell^1: ||z - e_i|| = ||z + e_i||\} = \{z \in \ell^1: z(i) = 0\}$, a closed hyperplane. As $X = E(-e_i,e_i) \oplus [e_i]$, $E(-e_i,e_i)$ is a Chebyshev subspace if e_i admits a unique nearest point in $E(-e_i,e_i)$. Let $z \in E(-e_i,e_i)$ be a nearest point of e_i. Since $\theta \in E(-e_i,e_i)$ we have $||z-e_i|| \leq ||e_i|| = 1$ which is the same as $1 + \sum_{j \neq i} |z(j)| \leq 1$, because $z(i) = 0$. This shows that $z(j) = 0$ for all j and hence $z = \theta$. Thus θ is the unique nearest point of e_i and hence $E(-e_i,e_i)$ is Chebyshev. Clearly, e_i is an extreme point of the unit ball of ℓ^1. Also, if we write $M_i = [e_i]$, then $M_i^\theta = E(-e_i,e_i)$.

5.4 Nature of equidistant sets in ℓ^p Spaces. A normed linear space X is said to have the

1) P_1- property if for all $x,y \in X$, $E(x,y)$ is weakly closed,

2) P_2- property if for each $x \in X$ with $||x|| = 1$, there exists $\varepsilon_x > 0$ such that whenever y and z are distinct points of the set $x + \varepsilon_x U(X)$, then the intersection $E(y,z) \cap (\varepsilon_x U(X))$ is weakly closed.

That there is a connection between P_1 and P_2 properties and the continuity behavior of metric projections onto Chebyshev sets is indicated by a result of Klee ([52] , Proposition 2.5). Not much is known about spaces having the P_1-property. Apart from the finite dimensional and inner-product spaces, no other example of spaces possessing the P_1-property has appeared in literature. In the following we shall show that each equidistant set in an ℓ^p-space $(1 < p < \infty)$ is closed in the bounded weak topology and hence each ℓ^p-space $(1 < p < \infty)$ satisfies the P_2-property. Since $E(y,z)$ can be brought to the form $E(-x,x)$ by a suitable translation we need only consider the latter type of sets. First, we prove a simple inequality.

Lemma 5.4.1. Let $p \geq 1$ and y and z be any two complex numbers. Then the following inequality holds:

$$||y+z|^p - |y-z|^p| \leq 2^p\, p(|y^{p-1}z| + |z|^p).$$

Proof : By triangle inequality $|y+z|^p \leq (|y|+|z|)^p$ and $|y-z|^p \geq ||y|-|z||^p$ and hence

$$|y+z|^p - |y-z|^p \leq (|y|+|z|)^p - ||y| - |z||^p \qquad (5.4.2)$$

Since the R.H.S. of (5.4.2) is unaffected by changing the sign of z we obtain

$$||y+z|^p - |y-z|^p| \leq (|y|+|z|)^p - ||y| - |z||^p \qquad (5.4.3)$$

Thus we need only prove the lemma in the form :

$$(|y| + |z|)^p - ||y| - |z||^p \leq 2^p\, p(|y^{p-1}z| + |z^p|) \qquad (5.4.4)$$

Case I. Let $|z| \leq |y|$. If we write $x = |z/y|$, then (5.4.4) reduces to

$$(1+x)^p - (1-x)^p \leq 2^p\, p(x+x^p),\ 0 \leq x \leq 1 \qquad (5.4.5)$$

Set $F(x) = (1+x)^p - (1-x)^p - 2^p p\,(x+x^p)$, $0 \leq x \leq 1$. Then $F(0) = 0$, $F(1) = 2^p - 2^{p+1}p = 2^p(1-2p) < 0$, and

$$F'(x) = p[(1+x)^{p-1}+(1-x)^{p-1}-2^p(1+px^{p-1})] \leq p\,[\max_{0\leq x\leq 1} \{(1+x)^{p-1} +$$

$$(1-x)^{p-1}\} - 2^p \min_{0\leq x\leq 1} (1+px^{p-1})] \leq p[2^{p-1}+2^{p-1}-2^p]=0 \text{ for } 0 \leq x \leq 1$$

Hence $F(x) \leq 0$ for $0 \leq x \leq 1$ and this is precisely the relation (5.4.4).

Case II. Let $|y| < |z|$. In this case we take $x = |y/z|$ and

(5.4.4) then reduces to

$$(1+x)^p - (1-x)^p \leq 2^p\, p(1+x^{p-1}), \quad 0 \leq x \leq 1 \tag{5.4.6}$$

This inequality, however, follows directly from (5.4.5). Hence the lemma is proved.

Next we prove a variant of Lebesgue's Dominated Convergence Theorem for ℓ^1. This will be used to prove the main result of this section.

<u>Lemma</u> 5.4.2. Let $\{\phi_\alpha , D\}$ be a net in ℓ^1 converging pointwise to ϕ. If there exists a net $\{f_\alpha, D\}$ in ℓ^1 which converges in norm to an element f and if $|\phi_\alpha| \leq f_\alpha$ for every $\alpha \in D$, then $\phi \in \ell^1$ and $\sum\limits_{i=1}^{\infty} \phi_\alpha(i) \to \sum\limits_{i=1}^{\infty} \phi(i)$.

<u>Proof</u> : Since $|\phi_\alpha(i)| \leq f_\alpha(i)$ for all $\alpha \in D$, taking limit we obtain $|\phi(i)| \leq f(i)$ and this is true for all i. Hence $\sum\limits_{i=1}^{\infty} |\phi(i)| \leq \sum\limits_{i=1}^{\infty} f(i) < \infty$ as $f \in \ell^1$ which shows that $\phi \in \ell^1$. Now

$$\begin{aligned}
\left|\sum_{i=1}^{\infty} \phi_\alpha(i) - \sum_{i=1}^{\infty} \phi(i)\right| &\leq \left|\sum_{i=1}^{i_o} \phi_\alpha(i) - \sum_{i=1}^{i_o} \phi(i)\right| + \left|\sum_{i=i_o+1}^{\infty} \phi_\alpha(i) - \sum_{i=i_o+1}^{\infty} \phi(i)\right| \\
&\leq \left|\sum_{i=1}^{i_o} \phi_\alpha(i) - \sum_{i=1}^{i_o} \phi(i)\right| + \sum_{i=i_o+1}^{\infty} |\phi_\alpha(i)| + \sum_{i=i_o+1}^{\infty} |\phi(i)| \\
&\leq \left|\sum_{i=1}^{i_o} \phi_\alpha(i) - \sum_{i=1}^{i_o} \phi(i)\right| + \sum_{i=i_o+1}^{\infty} f_\alpha(i) + \sum_{i=i_o+1}^{\infty} f(i)
\end{aligned}$$

$$= \left| \sum_{i=1}^{i_o} \phi_\alpha(i) - \sum_{i=1}^{i_o} \phi(i) \right| + \sum_{i=i_o+1}^{\infty} (f_\alpha(i)-f(i)) + 2 \sum_{i=i_o+1}^{\infty} f(i)$$

$$\leq \left| \sum_{i=1}^{i_o} \phi_\alpha(i) - \sum_{i=1}^{i_o} \phi(i) \right| + ||f_\alpha - f|| + 2 \sum_{i=i_o+1}^{\infty} f(i) \quad (5.4.7)$$

Given $\varepsilon > 0$, there exist $\gamma \in D$ and a sufficiently large i_o such that each of the summands in the R.H.S. of (5.4.7) is less than $\varepsilon/3$ for $\alpha \geq \gamma$. This is precisely what we required to prove and this completes the proof.

Remark 5.4.3 Taking $\phi_n = f_n = e_n/n$, where $e_n(m) = \delta_{nm}$ and observing that $\{\phi_n\}$ is not dominated by a single $f \in \ell^1$, we see that Lemma 5.4.2 could be applied in situations in which Lebesgue's Dominated Convergence Theorem does not help.

Lemma 5.4.4. Let X be a normed linear space, and let $\mathcal{B}$ a subset in X^* such that the closed linear span of $\mathcal{B}$ is the whole space X^*. If $\{x_\alpha\}$ is a bounded net in X and x is an element in X such that for every $f \in \mathcal{B}$, $f(x_\alpha) \to f(x)$, then $x_\alpha \rightharpoonup x$.

Proof: Let $||x_\alpha|| \leq K$ for all α, and let g be an arbitrary nonzero functional in X^*. Then given $\varepsilon > 0$, there exist elements $f_1, f_2, \ldots, f_n$ in $\mathcal{B}$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$||g - \sum_{i=1}^{n} \alpha_i f_i|| \leq \varepsilon/2\ (K + ||x||).$$

Now

$$|g(x_\alpha - x)| \leq |(g - \sum_{i=1}^{n} \alpha_i f_i)(x_\alpha - x)| + \sum_{i=1}^{n} |\alpha_i| \; |f_i(x_\alpha - x)|$$

$$\leq ||g - \sum_{i=1}^{n} \alpha_i f_i||(||x_\alpha|| + ||x||) + \sum_{i=1}^{n} |\alpha_i| \; |f_i(x_\alpha - x)|$$

$$\leq \varepsilon/2 + \sum_{i=1}^{n} |\alpha_i| \; |f_i(x_\alpha - x)| \qquad (5.4.8)$$

Since $f_i(x_\alpha) \to f_i(x)$ for $i = 1,2,\ldots,n$, there exists γ such that $\sum_{i=1}^{n} |\alpha_i| \; |f_i(x_\alpha - x)| \leq \varepsilon/2$ for $\alpha \geq \gamma$ and hence $|g(x_\alpha) - g(x)| \leq \varepsilon$ for $\alpha \geq \gamma$. As g is arbitrary $x_\alpha \to x$.

<u>Theorem</u> 5.4.5. Let x be any point of ℓ^p $(1 < p < \infty)$. Then $(-x,x)$ is closed in the bounded weak topology of the space.

<u>Proof</u>: Let $\{u_\alpha, D\}$ be a bounded net in $E(-x,x)$ converging weakly to u. Then

$$||u_\alpha - x|| = ||u_\alpha + x|| \text{ for all } \alpha \in D,$$

i.e. $$\sum_{i=1}^{\infty} [|u_\alpha(i) - x(i)|^p - |u_\alpha(i) + x(i)|^p] = 0 \qquad (5.4.9)$$

Let $$z_\alpha(i) = |u_\alpha(i) - x(i)|^p - |u_\alpha(i) + x(i)|^p,$$

$$z(i) = |u(i) - x(i)|^p - |u(i) + x(i)|^p,$$

$$w_\alpha(i) = 2^p p[|u_\alpha^{p-1}(i) \; x(i)| + |x(i)|^p],$$

$$w(i) = 2^p\, p[|u^{p-1}(i)\ x(i)| + |x(i)|^p],$$

$$g_\alpha(i) = ||u_\alpha^{p-1}(i)| - |u^{p-1}(i)||, \text{ and}$$

$$y(i) = |x(i)|.$$

Clearly, $z_\alpha, w_\alpha, z, z \in \ell^1$ and $z_\alpha \to z$ pointwise since $u_\alpha(i) \to u(i)$ for all i (weak convergence in $\ell^p (1 \leq p < \infty)$ implies pointwise convergence). By Lemma 5.4.1, we have

$$|z_\alpha(i)| \leq w_\alpha(i) \text{ for all } \alpha \in D. \tag{5.4.10}$$

The net $\{g_\alpha\}$ is in ℓ^q where $\frac{1}{p} + \frac{1}{q} = 1$ and

$$||g_\alpha|| = \left(\sum_{i=1}^{\infty} ||u_\alpha^{p-1}(i)|-|u^{p-1}(i)||^q\right)^{1/q} \leq \left(\sum_{i=1}^{\infty} |u_\alpha^{p-1}(i)|^q\right)^{1/q}$$

$$+ \left(\sum_{i=1}^{\infty} |u^{p-1}(i)|^q\right)^{1/q} = \left(\sum_{i=1}^{\infty} |u_\alpha^p(i)|\right)^{\frac{1}{q}} + \left(\sum_{i=1}^{\infty} |u^p(i)|\right)^{\frac{1}{q}} = ||u_\alpha||^{\frac{p}{q}} + ||u||^{\frac{p}{q}}.$$

Since $\{u_\alpha\}$ is a bounded net this shows that $\{g_\alpha\}$ is also bounded. Moreover, $g_\alpha(i) \to 0$ for all i. Since the set $B = \{e_i\}_{i=1}^{\infty}$ forms a Schauder base for ℓ^p $(1 < p < \infty)$, by Lemma 5.4.4, $g_\alpha \rightharpoonup \theta$. Thus

$$||w_\alpha - w|| = 2^p\, p \sum_{i=1}^{\infty} g_\alpha(i)\, y(i) = 2^p\, p \langle g_\alpha, y\rangle \to 0 \tag{5.4.11}$$

where $\langle g_\alpha, y\rangle$ represents the value of the bounded linear

functional $y \in \ell^p$ at $g_\alpha \in \ell^q$. Applying Lemma 5.4.2 to (5.4.10) and (5.4.11) we see that

$$\sum_{i=1}^{\infty} z_\alpha(i) \to \sum_{i=1}^{\infty} z(i) \tag{5.4.12}$$

But $\sum_{i=1}^{\infty} z_\alpha(i) = 0$ for all $\alpha \in D$. Hence $\sum_{i=1}^{\infty} z(i) = 0$ i.e. $||u-x||^p = ||u+x||^p$ and hence $||u-x|| = ||u+x||$. This proves the theorem.

<u>Remark</u> 5.4.6. Let x be an element of $\ell^p (1 \leq p < \infty)$ with finitely many nonzero coordinates. Then $E(-x,x)$ is weakly closed.

<u>Proof</u>: Enumerate the nonzero co-ordinates of x by the numbers $x(i_1)$, $x(i_2), \ldots, x(i_n)$. If $z_\alpha \in E(-x,x)$, then $||z_\alpha - x||^p = ||z_\alpha + x||^p$,

i.e. $$\sum_{j=1}^{n} |z_\alpha(i_j) - x(i_j)|^p + \sum_{j \neq 1,2,\ldots,n} |z_\alpha(i_j)|^p = \sum_{j=1}^{n} |z_\alpha(i_j) + x(i_j)|^p + \sum_{j \neq 1,2,\ldots,n} |z_\alpha(i_j)|^p,$$

or $$\sum_{j=1}^{n} |z_\alpha(i_j) - x(i_j)|^p = \sum_{j=1}^{n} |z_\alpha(i_j) + x(i_j)|^p,$$

the expression $\sum_{j \neq 1,2,\ldots,n} |z\ (i_j)|^p$ being finite. Therefore, if $z_\alpha \rightharpoonup z$ then because of finite number of summands in the above equation we have

$$\sum_{j=1}^{n} |z(i_j) - x(i_j)|^p = \sum_{j=1}^{n} |z(i_j) + x(i_j)|^p, \quad (5.4.13)$$

and adding the expression $\sum_{j \neq 1,2,\ldots,n} |z(i_j)|^p$ to both the sides of (5.4.13) we get the required result.

We do not know whether in Theorem 5.4.5 the bounded weak topology can be replaced by the weak topology or not. However, it is clear that each ℓ^p-space $(1 < p < \infty)$ satisfies the P_2-property. As a consequence we have the following ([52] , Proposition 2.5) :

Proposition 5.4.7. Let Q be a boundedly compact Chebyshev set in $\ell^p(1 < p < \infty)$. Then each point of $\ell^p \sim Q$ admits a neighbourhood on which the restricted metric projection is weakly continuous.

Proposition 5.4.8. Let M be a closed linear subspace of $\ell^p(1 < p < \infty)$; P_M the metric projection onto M. Then P_M is continuous both from the strong to strong topology, and from the bounded weak to bounded weak topology on ℓ^p.

Proof: Because of the uniform convexity of the ℓ^p-space, M is an approximatively compact Chebyshev set and hence supports a continuous metric projection (Singer [71]). To show that P_M is continuous in the bw-topology of ℓ^p, we observe that for each $x \in X$, and $M_x = [x]$, M_x^θ is bw-closed on account of Theorems 5.3.6 and 5.4.5. By Proposition 5.2.3 we have $M^\theta = \bigcap_{x \in M} M_x^\theta$

which shows that M^θ is bw-closed. The bw-continuity of P_M then follows from Proposition 5.2.4.

Remark 5.4.9. The above has been essentially observed by Holmes [37] by using the fact that an ℓ^p-space $(1 < p < \infty)$ has a weakly sequentially continuous duality mapping.

In the case of ℓ^1, since strong and weak sequential convergence coincide, $E(-x,x)$ is weakly sequentially closed for each x. This property of ℓ^1 is weaker than the corresponding property of other ℓ^p-spaces. It is interesting to note that this common property of ℓ^p-spaces distinguishes them from $L^p(\mu)$ spaces $(1 < p < \infty, p \neq 2)$, where μ is a separable non-atomic measure. Indeed, Lambert [38,56] has shown that M^θ is weakly sequentially dense for any finite dimensional Chebyshev subspace M and consequently, $E(-x,x)$ cannot be weakly sequentially closed. In the following we show that c_o is another example of a space where no equidistant set is weakly sequentially closed.

Theorem 5.4.10. Let x be any point in c_o. Then $E(-x,x)$ is not weakly sequentially closed.

Proof: Let $x = (x_1,x_2,\ldots,x_n,\ldots) \in c_o$. Take

$$z_n(i) = \begin{cases} 0, & \text{if } i \neq n \\ 2\|x\| \operatorname{sgn} x_n + x_n, & \text{if } i = n \text{ and } x_n \neq 0 \\ 2\|x\|, & \text{if } i = n \text{ and } x_n = 0 \end{cases}$$

Then $||z_n - x|| = 2||x||$ for all n and $||z_n - 2x|| = 2||x||$ for all sufficiently large n. Hence $z_n \in E(x,2x)$ eventually. Since the sequence $\{z_n\}$ is bounded and converges pointwise to θ, it also converges weakly to θ (see Kothe [55], page 324). But $\theta \notin E(x,2x)$ and hence $E(x,2x)$ cannot be weakly sequentially closed. The result then follows from the relation $E(-x,x) = 2E(x,2x) - 3x$.

Corollary 5.4.11. No one dimensional Chebyshev subspace of c_o can have a weakly sequentially continuous metric projection.

Proof: Let $M = [x]$ be Chebyshev, and $\{z_n\}$ be the sequence described in Theorem 5.4.10. Then by Lemma 5.3.5, $P_M(z_n) \in \{tx: 1 \leq t \leq 2\}$ for sufficiently large n, and $P_M(\theta) = \theta$. So $P_M(z_n) \not\to \theta$ which shows that P_M is not weakly sequentially continuous.

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